Implications of implicit credit spread volatilities on interest rate modelling

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Abstract

We test seven term structure models in the Heath-Jarrow-Morton (1992) class in order to find the best representation of the Libor rate in interest rate markets after the credit crunch of 2007. The Libor rate is considered as a risky rate, subject to the credit risk of a generic counterparty whose credit quality is refreshed at each fixing date. We study the volatilities of the credit spreads implicitly obtained from Libor time series. In order to understand how assumed volatility functions affect interest rate curve modelling and asset pricing, we develop a model to estimate basis swap prices through the Monte Carlo simulations. We compare obtained results and individuate systematic relations existing between the basis spread forecast error and both the accuracy in volatility modelling and the accuracy of the Monte Carlo method. We analyze and document these relations by defining appropriate pricing error measures.

Keywords: Finance; Arbitrage-free models; Libor; Term structure; Volatility modelling.

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1 Introduction

A number of stylized facts about anomalies arisen in the interest rate market after the credit crunch of summer 2007 have been uncovered in the literature. Some examples are the appearance of basis spreads between interest rates with different tenors, the loss of the possibility of pricing swaps by using market spot rates, and the fact that the interest rate curve underlying of interest rate derivatives does not coincide with the discounting interest rate curve anymore. However, the most relevant consequence of the credit crisis is that a spread has opened up between the Libor rates and the risk-free Eonia OIS rates, Overnight Indexed Swaps rates. This event has led to a new mathematical modelling of the Libor rate as a risky interest rate. The credit risk of the Libor rate can be measured by a credit spread that is not referred to a specific counterparty, but to a generic one whose credit quality is refreshed at each fixing date. The fixings are trimmed averages of contributions from a panel of the most relevant banks in the market with the highest credit quality.

Although many authors have proposed new approaches for modeling the Libor rate (see Mercurio (2009), Ametrano and Bianchetti (2009), Henrard (2010), Pallavicini and Tarenghi (2010), Morini (2011),Crépey et al. (2012), Bianchetti and Morini (2013), Pallavicini and Brigo (2013), Crépey et al. (2014), Grbac and Runggaldier (2015), and Fanelli (2016)), to the best of our knowledge and to date, there is no published paper that focuses on credit spread volatility modelling and its impact on interest rate derivative pricing. In this paper, we aim at filling this gap in the literature. We investigate the characteristics of the credit spread volatility and we test seven specific term structure models for credit spreads in the Heath Jarrow Morton (1992) (HJM, henceforth) class, that use seven different volatility functions as inputs. From market data we obtain the daily term structure of forward credit spreads, defined through the implied default intensity of the contributing banks of the Libor corresponding to a chosen tenor. Furthermore, we evaluate the implicit credit

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spread volatilities. For each model, we use these data to estimate the volatility function parameters. In order to assess the accuracy of models in representing the behaviour of the credit spread, we test their ability to predict the price of an interest rate derivative, the basis swap. We document systematic discrepancies between the various models and market prices, as a function of the accuracy of the chosen volatility model and the accuracy of the Monte Carlo method. Finally, we identify and quantify the effect of the implicit volatility modelling on the accuracy of the basis spread pricing model. Based on these analyses, we provide some recommendations regarding the type of models that should be used in research and practice.

The HJM approach offers several advantages in modelling term structures in the arbitrage-free environment (see Heath et al. (1992), Brigo and Mercurio (2006), and Bielecki and Rutkowski (2000)). The HJM model is automatically calibrated to the initial yield curve. As a results, claim prices are completely determined by a description of the volatility structure of interest rate changes. In particular, the drift term is a function of the volatility, so that estimates of expected rate changes are not needed. In addition, it is possible to have different HJM models choosing different volatility functions, also path-dependent, giving the possibility to make the model consistent with the real market situation. We exploit these features of the HJM approach to investigate the implications of volatility function modelling on interest rate derivatives, through the definition of seven different volatility functions.

Much academic research has dealt with forward rate volatility specifications which give rise to HJM models, often path-dependent and multi dimensional. Among the most relevant papers we recall Amin and Morton (1994), Trolle and Schwartz (2009), Chiarella et al. (2004), and Moreni and Pallavicini (2014). Amin and Morton (1994) are the pioneers in testing the HJM model according to different volatility functions. They study the time series of implied interest rate volatilities from the HJM models and price options in order to investigate the accuracy of the proposed models. Trolle and Schwartz (2009) develop a tractable and flexible multifactor model of interest rate term structure. Among features of the model they consider unspanned stochastic volatility factors and correlation between innovations to forward rates and their volatilities. The authors show the model has a very good fit to bonds and interest rate derivatives. Chiarella et al. (2004) derive classes of interest rate models resembling the traditional models from the HJM framework, with the ultimate goal being the development of a unifying framework, or technique, capable of generating other models in a systematic manner. Moreni and Pallavicini (2014) propose a parsimonious model based on observed rates that deduces yield-curve dynamics from a single family of Markov processes. They calibrate the model with two driving factors and deterministic volatility to at-the-money swaption prices to size the effect of the new degree of freedom introduced to model different tenors.

In a great deal of literature, the HJM model has also been extended in order to model defaultable interest rate and price credit risk derivatives. We review some articles. Duffie and Singleton (1999) provide a discrete-time reduced form model in order to evaluate risky debt and credit derivatives in an arbitrage-free environment. They add a forward spread process to the forward risk-free rate process and use the HJM approach to obtain the arbitrage-free drift restriction. Collin-Dufresne et al. (2004) demonstrate a pricing formula for defaultable securities, when the no-jump condition is violated. They introduce a new probability measure under which computing the expectation of claim cash flows. Henrard (2010) and Pallavicini and Tarenghi (2010) propose two different frameworks to construct yield curves consistent with a multi-curve situation and derive the price of interest rate derivatives. Chiarella et al. (2011) develop a simulation approach for defaultable yield curves. The default event is modelled using the Cox process where the stochastic intensity represents the credit spread. The forward credit spread volatility function is affected by the entire credit spread term structure. They provide the defaultable bond and credit default swap option price in a probability setting equipped with a subfiltration structure. Crépey et al. (2012) apply a defaultable HJM approach to model the term structure of multiple interest rate curves. They choose a class of non-negative multidimensional Lévy processes as driving processes combined with deterministic volatility structures, in order to obtain a flexible and efficient interest rate derivative pricing model. Eberlein and Grbac (2013) model credit risk within the LMM. They propose a rating Lévy Libor model that is arbitrage-free for defaultable forward Libor rates related to risky bonds with credit ratings. They use time-inhomogeneous Lévy processes as driving processes. Recently, Pallavicini and Brigo (2013) model multiple LIBOR and OIS based interest rate curves consistently, based only on market observables and by consistently including credit, collateral and funding effects. They develop a framework for pricing collateralized interest-rate derivatives. Crépey et al. (2014) develop a parsimonious Markovian multiple-curve model for evaluating interest rate derivatives in the post-crisis setup and they use BSDE-based numerical computations for obtaining counterparty risk and funding adjustments. Cuchiero et al. (2016) propose a general semimartingale framework for modeling multiple yield curves which have emerged after the last financial crisis. They use a HJM approach to model the term structure of multiplicative spreads between FRA rates and simply compounded OIS risk-free forward rates under a risk-neutral measure. They show that the proposed framework allows to unify and extend several recent approaches to multiple yield curve modeling. Fanelli (2016) uses a defaultable HJM methodology to model the term structure of the credit spread, defined implicitly in the Libor. A forward credit spread volatility function depending on the entire credit spread term structure is assumed and a model for basis swaps is proposed.

We extend the model proposed in the work of Fanelli (2016) to consider a class of defaultable HJM models in order to analyze the impact of volatility modelling on interest rate curve estimation through the pricing of a basis swap.

The paper proceeds as follows. In Section 2, we describe the HJM approach used to model the risky Libor rate. In Section 3, we define seven implicit credit spread volatility functions and estimate their parameters. In Section 4, we illustrate the model used to price a basis swap and document the biases we find between model and market data. Section 5 deals with the analysis of the implications of volatility modelling on the pricing model. Finally, Section 6 concludes.

2 The defaultable HJM approach

We assume a finite time horizon $\overline{T}$, where the uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. $\mathcal{F} = \mathcal{F}_{\overline{T}}$ is the σ-algebra at time $\overline{T}$. All statements and definitions are understood to be valid until the time horizon $\overline{T}$.

Following Fanelli (2016), let $\mathbf{H} = (\mathcal{H}_t)_{t\geq 0} = (\sigma(X_s : 0 \leq s \leq t))_{t\geq 0}$ be the filtration generated by the background driving process X, that is an $\mathbb{R}^d$-valued right continuous stochastic process $X = \{X_t : 0 \leq t \leq \overline{T}\}$ with left limit. It represents the flow of all background information and $\mathcal{H} = \mathcal{H}_{\overline{T}}$ is the sub-σ-algebra at time $\overline{T}$. Then, we consider a generic counterparty C^z of a lending contract at time z, that defaults at time $\tau^z > z$. Let τ_i^z, $i \in \mathbb{N}$, be the times of default generated by a Cox process $N^z(t) = \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i^z \leq t\}}$ associated to C^z. We only consider the time of the first default and it will be referred to with $\tau^z := \tau_1^z$. Then, the time τ^z is a stopping time, $\tau^z : \Omega \to [0, +\infty[$, defined as the first jump time of the Cox process $N^z(t)$, that is

$$\tau^z = inf\{t \geq 0 | N^z(t) > 0\}.$$

The right-continuous default indicator process $\mathbf{1}_{\{\tau^z \leq t\}}$ generates the generic subfiltration $\mathbf{F}^{\tau^z} = (\mathcal{F}_t^{\tau^z})_{t\geq 0} = (\sigma(\mathbf{1}_{\{\tau^z \leq s\}} : 0 \leq s \leq t))_{t\geq 0}$. When we consider N counterparties

$C^1, C^2, ..., C^N$, $N \in \mathbb{N}$, we state $\mathbf{F}^{\tau} = \mathbf{F}^{\tau^1} \vee \mathbf{F}^{\tau^2} \vee ... \vee \mathbf{F}^{\tau^N}$. The filtration $\mathbf{H}$ is enlarged to obtain $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$, such that $\mathbf{F} = \mathbf{H} \vee \mathbf{F}^{\tau}$, which is $\mathcal{F}_t = \mathcal{H}_t \vee F^{\tau}_t$, $\forall t \geq 0$. Since obviously $\mathcal{F}^{\tau^z}_t \subset \mathcal{F}_t$, $\forall t \geq 0$, τ^z is a stopping time with respect to $\mathbf{F}$, but it is not necessarily a stopping time with respect to $\mathbf{H}$. The right-continuous stochastic process $\lambda^z(t)$ is the intensity of the Cox process. It is independent of $N^z(t)$, it is assumed to be adapted to $\mathbf{H}$, and follows the diffusion process

$$d\lambda^z(t) = \mu^z_{\lambda}(t)dt + \sigma^z_{\lambda}(t)dW^z_{\lambda}(t),$$

where $\mu^z_{\lambda}(t)$ is the drift of the intensity process, $\sigma^z_{\lambda}(t)$ is the volatility of the intensity process and W^z_{λ} is a standard Wiener process under the objective probability measure $\mathbb{P}$. Processes W^z_{λ}, $z = 1, ..., N$, are assumed to be N independent Wiener processes.

We define the defaultable instantaneous forward rate, $f^z(t, T)$, $0 \leq t \leq T \leq \overline{T}$, as the sum of the risk-free instantaneous forward rate, $f(t, T)$, and the instantaneous forward credit spread $\lambda^z(t, T)$, so that we have

$$f^z(t, T) := f(t, T) + \lambda^z(t, T). \tag{1}$$

Thus, the forward credit spread is obtained as difference between the two forward interest rates. If $t = T$, then we obtain the defaultable spot rate $f^z(t) := f^z(t, t) = r(t) + \lambda^z(t)$, where $r(t) := f(t, t)$ represents the risk-free spot rate and $\lambda^z(t) := \lambda^z(t, t)$ is the spot credit spread. The credit spread is referred to as the Cox intensity across maturities (see Jeanblanc and Rutkowski (2002), Jamshidian (2004) and Brigo and Morini (2005)).

In the HJM framework the term structure of risk-free forward rates are modelled according to the following stochastic integral equation

$$f(t, T) = f(0, T) + \int_0^t \mu(v, T)dv + \int_0^t \sigma_f(v, T)dW(v), \tag{2}$$

where $\mu(t,T)$ is the instantaneous forward rate drift function, $\sigma_f(t,T)$ is the instantaneous forward rate volatility function and $W(t)$ is a standard Wiener process with respect to the objective probability measure $\mathbb{P}$. Both the drift and the volatility are allowed to be stochastic processes and not necessarily deterministic functions.

We also define the dynamics of instantaneous credit spreads $\lambda^z(t,T)$ as follows:

$$\lambda^z(t,T) = \lambda^z(0,T) + \int_0^t \mu_\lambda^z(s,T)ds + \int_0^t \sigma_\lambda^z(s,T)dW_\lambda^z(s). \tag{3}$$

Again, both the drift and the volatility are allowed to be stochastic processes and not necessarily deterministic functions.

Using the HJM approach, we apply the the no-arbitrage restriction on the drifts to dynamics (2) and (3). So we find the following forward dynamics, respectively for the risk-free forward rate and the forward credit spread, under the risk neutral probability measure $\widetilde{\mathbb{P}}$:

$$f(t,T) = f(0,T) + \int_0^t \sigma_f(v,T)\int_v^T \sigma_f(v,s)dsdv + \int_0^t \sigma_f(v,T)d\widetilde{W}(v),$$

and

$$\begin{aligned}\lambda^z(t,T) \;=\; & \lambda^z(0,T) + \int_0^t \sigma_\lambda^z(v,T)\int_v^T \sigma_\lambda^z(v,s)dsdv + \\ & \int_0^t \rho\left[\sigma_f(v,T)\int_v^T \sigma_\lambda^z(v,s)ds + \sigma_\lambda^z(v,T)\int_v^T \sigma_f(v,s)ds\right]dv + \\ & \int_0^t \sigma_\lambda^z(v,T)d\widetilde{W}_\lambda^z(v),\end{aligned} \tag{4}$$

where ρ is the correlation coefficient between the two Wiener processes $\widetilde{W}(t)$ and $\widetilde{W}_\lambda^z(t)$ under the risk neutral probability measure and we assume they are one-dimensional (see Chiarella et al. (2011) for further mathematical details in calculating the expression for the stochastic differential equation (4)).

Furthermore, we define the generic spot Libor rate $L(t, t+\alpha)$ as a simply compounded interest rate fixed at time t for the time interval $[t, t+\alpha]$, and it is given by

$$L(t, t+\alpha) = \left(\frac{1}{P^z(t, t+\alpha)} - 1 \right) \frac{1}{\alpha}, \tag{5}$$

where α represents the tenor, that is the difference between two successive fixing dates, $P^z(t, T)$ is the defaultable zero-coupon bond price of C^z, the counterparty in the Libor market at time z, and we assume zero recovery rate. We state that the credit risk associated to the Libor rate is not referred to a specific counterparty, but a generic one, whose credit quality is "refreshed" at each fixing date. Moreover, the shorter the tenor, the lower the credit risk. Since C^z is a market counterparty at time z, it necessarily follows that $\tau^z > z$, so we have that $P^z(t, T)$ in (5) is the pre-default bond price, such that

$$P^z(t, T) = e^{-\int_t^T (f(t,s) + \lambda^z(t,s)) ds}. \tag{6}$$

Instead, the generic price of the defaultable zero-coupon bond is

$$P^z(t, T) = \mathbf{1}_{\{\tau^z > t\}} e^{-\int_t^T f^z(t,s) ds}.$$

Through formulas (5) and (6) we can state that instantaneous yield curves $f(t, T)$ and $\lambda^z(t, T)$ are implicitly defined by the Libor rate. However, we are not considering a single HJM model, but multiple HJM models simultaneously, since Libor rates refer to different counterparties at different fixing times, whose default times are driven by different stochastic processes. Furthermore, the no-arbitage conditions are also different.

In the market, the standard assumption used to be

$$\lambda^z(0,T) = \lambda^0(0,T), \tag{7}$$

but this is no longer valid after the credit crunch. Indeed, we have to take into account that $\lambda^z(t,T)$ is the credit spread of C^z, that has the credit quality of a Libor counterparty at time z, while $\lambda^0(t,T)$ is the credit spread of C^0, that has the credit quality of a Libor counterparty at time 0. After the credit crunch, C^0 at z could have a credit quality lower than C^z. We recall that the credit risk associated to the Libor rate is referred to a generic counterparty, whose credit quality is "refreshed" at each fixing date. Therefore, since C^z has the credit quality "refreshed" at time z after 0, according to Fanelli (2016), we assume instead of (7) that

$$\lambda^z(0,T+z) = \lambda^0(0,T). \tag{8}$$

If we start with $z = \alpha$ in (8), where α is the tenor, the generic intensity process is $\lambda^\alpha(t,t+\alpha)$ with $0 \leq t < \alpha$ and with $\lambda^\alpha(0,t+\alpha) = \lambda^0(0,t)$. Then, at $t = \alpha$ we set $\lambda^{2\alpha}(\alpha,t+2\alpha) = \lambda^\alpha(\alpha,t+\alpha)$, $0 \leq t < \alpha$, and the process $\lambda^{2\alpha}(t+\alpha,t+2\alpha)$ is valid until $t = \alpha$, that is until the time 2α. And so on. In this way we can construct the term structure of credit spreads.

For example, for the initial counterparty C^0, we have that $\lambda^0(t,T)$ is given by (4) with $z = 0$. However, when $z = \alpha$, $\lambda^\alpha(\alpha,t+\alpha)$, $0 \leq t < \alpha$, is given by

$$\begin{aligned}\lambda^\alpha(\alpha,t+\alpha) &= \underbrace{\lambda^\alpha(0,t+\alpha)}_{=\lambda^0(0,t)} + \int_0^\alpha \sigma_\lambda^\alpha(v,t+\alpha)\int_v^{t+\alpha}\sigma_\lambda^\alpha(v,s)\,ds\,dv + \\ &\int_0^\alpha \rho\left[\sigma_f(v,t+\alpha)\int_v^{t+\alpha}\sigma_\lambda^\alpha(v,s)ds + \sigma_\lambda^\alpha(v,t+\alpha)\int_v^{t+\alpha}\sigma_f(v,s)ds\right]dv + \\ &\int_0^\alpha \sigma_\lambda^\alpha(s,t+\alpha)d\widetilde{W}_\lambda^\alpha(v),\end{aligned} \tag{9}$$

where we can consider the following inequality

$$\sigma_\lambda^\alpha(v,s) \neq \sigma_\lambda^0(v,s) \text{ for } 0 \leq s < \alpha,$$

and in this paper we set, for a generic z, $0 \leq s < z$, that $\sigma_\lambda^z(v,s) = 0$.

We recall the main advantages of the HJM model are that in the formulation of the spot rate process and bond price process the market price of interest rate risk drops out by being incorporated into the Wiener process under the risk neutral measure. Furthermore, the model is automatically calibrated to the initial yield curve and the drift term in the forward rate differential equation is a function of the volatility term. The HJM approach to interest rate modeling ensures that claim prices are determined through volatility parameters, not through drift and premia. In addition, it is possible to have different HJM models choosing different volatility functions, that must describe the stochastic evolution of the entire term structure curve.

We have chosen the following seven forms for the volatility, with σ_0, σ_1, β, μ, v, $k \in \mathbb{R}$[1]:

1. Absolute (Ho and Lee (1986) model): $\sigma_\lambda^z(t,T) = \sigma_0$;
2. Square root: $\sigma_\lambda^z(t,T) = \sigma_0\sqrt{\lambda^z(t,T)}$;
3. Proportional: $\sigma_\lambda^z(t,T) = \sigma_0\lambda^z(t,T)$;
4. Linear Absolute: $\sigma_\lambda^z(t,T) = \sigma_0 + \sigma_1(T-t)$;
5. Exponential Proportional: $\sigma_\lambda^z(t,T) = (\sigma_0 + \sigma_1\lambda(t,T))e^{\beta(T-t)}$;
6. Linear Proportional: $\sigma_\lambda^z(t,T) = [\sigma_0 + \sigma_1(T-t)]\lambda^z(t,T)$;
7. Jump Stochastic: $\sigma_\lambda^z(t,T) = \sqrt{V(t)}$, where $(V(t))_{t\geq 0}$ follows the dynamic $dV(t) = \mu V(t)dt + vV(t)dW^V(t) + kdJ(t)$.

[1] Volatility functions 1, 2, 3, 4, and 6 have already been tested in a risk-free HJM framework by Amin and Morton (1994).

3 Estimation of implicit volatility functions

Our data set consists of daily time series, spanning from January 2007 to the half of May 2010, of the Eonia OIS rates and the Libor rates with monthly maturity, from one month to twelve months. Eonia OIS rates according to different maturities give the risk-free term structure, because the OIS rate with a generic maturity T is seen as an average of the market expectation of the overnight futures rates until T, and those rates are considered free of credit risk. On the contrary, Libor rates determine the defautable interest rate curve.

Since in the HJM framework we model the evolution of the forward interest rate curve, we first need to estimate the forward interest rates. Through formula (5) applied to the Libor rates, we can obtain every day the sequence of defaultable bonds (6) according to the twelve months as maturities. By interpolating the instantaneous forward interest rate curve across available maturities, we can then have a good approximation of the defaultable entire term structure. At each date t, we obtain the 12-dimensional vector of forward rates, $\boldsymbol{\psi}_t = [f(t,T_1), f(t,T_2), ..., f(t,T_{12})]$, where the maturity dates $T_1, ..., T_{12}$ are the months of the year. The term structure is completely defined by the vector $\boldsymbol{\psi}_t$, since we assume that the forward rates of all intermediate maturities are obtained by interpolating the forward rates for missing maturity dates in $\boldsymbol{\psi}_t$.

We calculate annual volatilities over each month, and we obtain the term structure of volatilities represented by the 12-dimensional vector $\boldsymbol{\Phi}_t = [\sigma^z_\lambda(t,T_1), \sigma^z_\lambda(t,T_2), ..., \sigma^z_\lambda(t,T_{12})]$, at a generic month t. Volatility data $\boldsymbol{\Phi}_t$ are used to estimate the parameters of the different volatility functions by using Nonlinear Least Squared Regression.

Absolute volatility. The absolute volatility is a constant function over time and across maturities and the HJM model coincides with the continuous-time interest rate curve model of Ho and Lee (1986). The constant function is:

$$\sigma_\lambda^z(t,T) = \sigma_0, \tag{10}$$

where σ_0 is estimated as the average annual volatility across maturities. It is equal to 0.891%. In this case, volatility term structure is flat and we also assume that the credit quality of the generic Libor counterparty remains constant over time, equal to that measured at inception. The model standard error is considered equal to the standard deviation of effective volatilities, that is 0.000646.

Square root volatility. The square root volatility function allows term structure depends on the credit quality of the generic counterparty at the generic date t. The function is:

$$\sigma_\lambda^z(t,T) = \sigma_0\sqrt{\lambda^z(t,T)}, \tag{11}$$

where σ_0 is the instantaneous volatility coefficient and the term $\sqrt{\lambda^z(t,T)}$ represents the dependence of the volatility from the level of the forward credit spread. We can say that this volatility is level-dependent. In Table 1, we show the results of the nonlinear regression.

Table 1: Parameter estimates from square root volatility model

	Estimate	Std Error	t statistic	p-value
σ_0	0.114242	0.00422704	27.0264	0.0000

Mean	0.008910	Std Deviation	0.000646
Squared residuals sum	0.000014	Std Error	0.001136

Proportional volatility. The proportional volatility is a level-dependent function, depending on the credit spread level. The function is:

$$\sigma_{\lambda}^{z}(t,T) = \sigma_0 \lambda^{z}(t,T), \tag{12}$$

where σ_0 is the proportionality coefficient, that expresses the proportional dependence on the credit spread. Table 2 contains the results of the regression analysis.

Table 2: Parameter estimates from proportional volatility model

	Estimate	Std Error	t statistic	p-value
σ_0	1.41955	0.0874596	16.2309	0.0000

Mean	0.008910	Std Deviation	0.000646
Squared residuals sum	0.000038	Std Error	0.001868

Linear absolute volatility. The linear absolute volatility is a deterministic function, depending on the time-to-maturity. The function is:

$$\sigma_{\lambda}^{z}(t,T) = \sigma_0 + \sigma_1(T-t), \tag{13}$$

where σ_0 represents the short-term coefficient, whereas σ_1 is the time-to-maturity proportionality coefficient. The regression results are shown in Table 3.

Table 3: Parameter estimates from linear absolute volatility model

	Estimate	Std Error	t statistic	p-value
σ_0	0.00800281	0.000261130	30.6468	0.0000
σ_1	0.000139638	3.54806e–005	3.9356	0.0028

Mean	0.008910	Std Deviasion	0.000646
Squared residuals sum	1.80e–06	Std Error	0.000424

Exponential proportional volatility. The exponential proportional volatility is a function of the time-to-maturity and of the credit spread level, and is defined as:

$$\sigma_\lambda^z(t,T) = (\sigma_0 + \sigma_1 \lambda(t,T))e^{\beta(T-t)}, \tag{14}$$

where σ_0 represents the short-term coefficient, σ_1 determines the size of the hump of the volatility curve, and β is the time-to-maturity proportionality coefficient. At each time t, the volatility depends on the forward credit spread itself, in accordance with the exponential time-to-maturity dependent factor $e^{\beta(T-t)}$. In Table 4, we display the nonlinear regression estimates.

Table 4: Parameter estimates from exponential proportional volatility model

	Estimate	Std Error	t statistic	p-value
σ_0	0.00989186	0.000214889	46.0325	0.0000
σ_1	-0.471824	0.0499961	-9.4372	0.0000
β	-0.0362326	0.00270461	-13.3966	0.0000
Mean		0.008910	Std Deviation	0.000646
Squared residuals sum		1.65e–07	Std Error	0.000136

Linear proportional volatility. The linear proportional volatility is a combination of functions (12) and (13), so that it is a level-dependent function, because its value depends on the credit spread level according to a proportionality coefficient. This proportionality coefficient is a deterministic term because it depends on the time-to-maturity. Then, the volatility function assumes the following form:

$$\sigma_\lambda^z(t,T) = [\sigma_0 + \sigma_1(T-t)]\lambda^z(t,T), \tag{15}$$

where σ_0 represents the short-term coefficient, and $\sigma_1(T-t)$ is the deterministic coefficient. The regression results are summarized in Table 5.

Table 5: Parameter estimates from linear proportional volatility model

	Estimate	Std Error	t statistic	p-value
σ_0	0.00935279	0.000694207	13.4726	0.0000
σ_1	−1.20733	0.101375	−11.9095	0.0000
Mean		0.008910	Std Deviation	0.000646
Squared residuals sum		0.000011	Std Error	0.001061

Jump stochastic volatility. The jump stochastic volatility function is defined as follows:

$$\sigma_\lambda^z(t,T) = \sqrt{V(t)}, \tag{16}$$

where $V(t)$ indicates the squared volatility of the credit spread and the process $(V(t))_{t\geq 0}$ follows the following stochastic dynamics with jumps:

$$dV(t) = \mu V(t)dt + vV(t)dW^V(t) + kdJ(t), \tag{17}$$

where $W^V(t)$ is the standard Brownian motion, such that $(W^\lambda(t))_{t\geq 0}$ and $(W^V(t))_{t\geq 0}$ are independent processes. $(J(t))_{t\geq 0}$ is the jump process, a discrete time process, i.e. compound Poisson process. The annualized frequency of jumps is given by the average of jumps per year, that we call ξ. The jump return size is k, which is determined by natural logarithm of the jump returns being normally distributed, $\ln(1+k) \sim N(\ln(1+\overline{k}) - 1/2\gamma^2, \gamma^2)$, where $\overline{k}$ is the mean jump size and γ is the standard deviation. Finally, $\mu = (r - \xi\overline{k})$ is the drift term and v is the diffusion coefficient. We use a jump-recursive filter estimation procedure to obtain parameters of equation (17). This procedure attempts to identify and characterise the lower frequency, the higher mean jump size and jump volatility. It consists of the following steps:

1. Calculate sample volatility standard deviations;
2. Identify volatilities greater than a fixed threshold, typically three or four standard data deviations, but it depends on market data; we call these volatilities "jumps";
3. Remove the jumps from the sample;
4. And repeat until convergence in jump intensity ξ, mean jump size $\overline{k}$ and jump volatility v.

In Table 6, we show the results of the jump-recursive filter estimation procedure.

Table 6: Parameter estimates from jump stochastic volatility model

Iteration	v	$\overline{v}^*$	No. Jumps	ξ	$\overline{k}$	γ
0	0.0068	0.0272	-	-	-	-
1	0.0047	0.0189	19	5.61	0.0023	0.0005
2	0.0040	0.0158	37	10.92	0.0018	0.0006
3	0.0035	0.0138	53	15.65	0.0016	0.0006
4	0.0031	0.0125	67	19.78	0.0015	0.0006
5	0.0029	0.0117	76	22.44	0.0014	0.0006
6	0.0029	0.0114	81	23.92	0.0014	0.0006
7	0.0028	0.0112	84	24.08	0.0013	0.0006
8	0.0028	0.0111	86	25.4	0.0013	0.0006
9	0.0028	0.0111	86	25.4	0.0013	0.0006

$^*\overline{v}$ is assumed to be equal to four times v

The model standard error of this model is considered equal to the standard deviation of effective volatilities, that is 0.000646.

Eonia rates. The relation between the Eonia rate and the instantaneous forward risk-free rate can be represented by the following formula, equivalent to the Libor representation (5):

$$E(t,T) = \left(\frac{1}{P(t,T)} - 1\right)\frac{1}{(T-t)} \tag{18}$$

where $P(t,T) = e^{-\int_t^T f(t,s)ds}$ is the risk-free bond at time t.

From (18) we obtain the instantaneous forward risk-free rates $f(t,T)$ according to different maturities, and we model the evolution of the term structure according to (2) and using the following volatility function:

$$\sigma_f(t,T) = \alpha f(t,T) e^{\delta(T-t)}. \tag{19}$$

Different volatility functional forms have been tested on empirical data, and the chosen volatility function is the one that, according to statistical tests, best fits historical data. The results of risk-free rate volatility parameter estimation on Eonia rate time series (January 2007 - May 2010) are summarized in Table 7.

Table 7: Results of risk-free rate volatility parameter estimation on Eonia rate time series (January 2007 - May 2010)

	Estimate	Std Error	t statistic	p-value
α	0.289229	0.00302311	95.6726	0.0000
δ	0.0399493	0.00127086	31.4349	0.0000
Mean		0.009466	Std Deviation	0.001465
Squared residuals sum		2.02e–07	Std Error	0.000142

Finally, from historical data the correlation coefficient ρ between $f(t,T)$ and $\lambda^\alpha(t,T)$ is also estimated and assumed constant over time and maturities. It is equal to 0.679, meaning that the two rates are positively correlated.

4 Basis swap pricing model

We aim at analyzing the impact of credit spread volatility on the term structure of forward credit spreads. We know from previous sections that credit spread term structure is implicitly determined by the Libor rates, therefore it results very difficult to find a way to measure the suitability of volatility models (10), (13), (12), (19), (15), (11), and (16), and study which model is the most consistent with the market data. Our test consists in measuring the error made in pricing a basis swap, according to the different volatility functions, in comparison with the real market price. We refer to the basis swap because it could be considered the plain vanilla interest rate derivative after the credit crunch.

We use a pricing model already adopted in Fanelli (2016), so that its effectiveness has already been proved. Following Morini (2009), if we consider a generic $\alpha/2\alpha$ Basis Swap, meaning that we exchange two swaps with the same fixed legs and pay two floating legs with frequencies respectively of α and 2α, we can calculate the price as the expectation of the leg cash flows discounted with riskless rate under the risk neutral measure $\widetilde{\mathbb{P}}$. The price of the $\alpha/2\alpha$ Basis Swap with maturity 2α, $P_{Basis}(0, 2\alpha, Z)$, is given by the following formula:

$$P_{Basis}(0, 2\alpha, Z) =$$

$$E_{\widetilde{\mathbb{P}}}[\underbrace{D(0,\alpha)\alpha(L(0,\alpha)+Z)+D(0,2\alpha)\alpha(L(\alpha,2\alpha)+Z)}_{\alpha-tenor\,leg}] - E_{\widetilde{\mathbb{P}}}[\underbrace{D(0,2\alpha)2\alpha L(0,2\alpha)}_{2\alpha-tenor\,leg}], \quad (20)$$

where Z is the Basis spread, given as the difference, in basis points, between the fixed rate of the higher frequency swap and the fixed rate of the lower frequency swap. In this paper, we adopt the convention that the Basis spread is added to the shorter tenor leg, and so the two fixed legs can be neglected in the Basis Swap pricing formula. $D(0,T) = e^{-\int_0^T r(s)ds}$ is the discount factor calculated using the risk-free spot rate $r(t)$. We use the risk-free spot rate since we consider interest rate derivatives that are collateralized.

Rearranging formula (20) we obtain

$$P_{Basis}(0,2\alpha,Z) = E_{\widetilde{\mathbb{P}}}[D(0,2\alpha)\alpha L(\alpha,2\alpha)] +$$
$$P(0,\alpha)\left(\frac{1}{P^0(0,\alpha)}-1\right) - P(0,2\alpha)\left(\frac{1}{P^0(0,2\alpha)}-1\right) + [P(0,\alpha)+P(0,2\alpha)]\alpha Z. \quad (21)$$

Pricing a Basis Swap means finding the basis spread Z that makes the quantity (21) equal to zero, namely

$$Z = \frac{P(0,2\alpha)}{\alpha[P(0,\alpha)+P(0,2\alpha)]}\left(\frac{1}{P^0(0,2\alpha)}-1\right) +$$
$$\frac{P(0,\alpha)}{\alpha[P(0,\alpha)+P(0,2\alpha)]}\left(1-\frac{1}{P^0(0,\alpha)}\right) - \frac{E_{\widetilde{\mathbb{P}}}[D(0,2\alpha)\alpha L(\alpha,2\alpha)]}{\alpha[P(0,\alpha)+P(0,2\alpha)]}. \quad (22)$$

In formula (22) all quantities, with the exception of the last expectation, can be simply calculated through equations (5) and (2) by using market data. Knowing $\lambda^\alpha(0,T)$ and the dynamics of $\lambda^\alpha(t,T)$ in equation (9), we can compute via Monte Carlo simulation the following expectation used in formula (22)

$$\begin{aligned} & E_{\widetilde{\mathbb{P}}}\left[D(0,2\alpha)\alpha L(\alpha,2\alpha)\right] \\ = \; & E_{\widetilde{\mathbb{P}}}\left[D(0,2\alpha)\left(\frac{1}{P^\alpha(\alpha,2\alpha)}-1\right)\right] \\ = \; & E_{\widetilde{\mathbb{P}}}\left[D(0,2\alpha)\left(e^{\int_\alpha^{2\alpha}(f(\alpha,s)+\lambda^\alpha(\alpha,s))ds}-1\right)\right]. \end{aligned} \quad (23)$$

We will use formula (22) to simulate seven basis swap models, according to the seven HJM models, and forecast the corresponding basis spreads. We will calculate errors of spread estimation with reference to the market quotation on 17^{th} May 2010 and we will compare the results according to the different HJM volatility models.

We can outline our forecasting procedure by the following points:

a) We choose the implicit volatility model;

b) From inception until basis swap maturity we simulate the daily evolution of the term structure of both the risk-free interest rates and credit spreads; we proceed by discretizing integrals in all the formulas with the Euler-Maruyama approximation technique (see Kloeden and Platen (1999));

c) We obtain the forecasted basis spread by applying the Monte Carlo method and according to different numbers of simulations;

d) We calculate two kinds of forecast errors: *i)* the *Absolute Error*, given by the difference between the true market spread observed at the valuation day and the forecasted value; *ii)* the *Relative Error*, computed as the absolute error divided by the market price.

The results obtained by applying the forecasting procedure are summarized in Table 8

Table 8

6/12 Basis Swap
Summary forecast procedure results and error measures.

This table reports, for seven different volatility functions: *(i)* results of estimated basis spreads Z (measured in basis points), standard deviations σ and standard errors $\hat{\sigma}$, according to different numbers Π of Monte Carlo simulations; *(ii)* measures of model forecast errors (measured in basis points), obtained as the deviation from the market price, *Abs Err*, and relative deviation from the market price, *Rel Err*.

				$Z_{market}^{5/17/2010} = 17\,bps$	
Volatility function	Z	σ	$\hat{\sigma}$	*Abs Err*	*Rel Err*
			$\Pi = 100,000$		
Absolute	17.129858	2.09492E-05	6.62472E-08	0.129858	0.007638706
Square root	17.131237	2.18688E-07	6.91553E-08	0.131237	0.007719824
Proportional	17.131889	2.24324E-05	7.09377E-08	0.131889	0.007758176
Linear absolute	17.238441	6.61367E-05	2.09142E-07	0.238441	0.014025941
Exponential linear	17.123505	1.77191E-05	5.60328E-08	0.123505	0.007265000
Linear proportional	17.115123	2.04307E-05	6.46075E-08	0.115123	0.006771941
Jump stochastic	17.14821	2.09043E-05	6.61054E-08	0.148210	0.008718235
			$\Pi = 10,000$		
Absolute	17.126594	2.09255E-05	2.09255E-07	0.126594	0.007446706
Square root	17.127936	2.17881E-05	2.17881E-07	0.127936	0.007525647
Proportional	17.119798	2.25518E-05	2.25518E-07	0.119798	0.007046941
Linear absolute	17.234835	6.60504E-05	6.60504E-07	0.234835	0.013813824
Exponential linear	17.119941	1.76597E-05	1.76597E-07	0.119941	0.007055353
Linear proportional	17.111019	2.05556E-05	2.05556E-07	0.111019	0.006530529
Jump stochastic	17.117345	2.09027E-05	2.09027E-07	0.117345	0.006902647
			$\Pi = 1,000$		
Absolute	17.224607	2.11943E-05	6.70224E-07	0.224607	0.013212176
Square root	17.229724	2.20564E-05	6.97487E-07	0.229724	0.013513176
Proportional	17.232824	0.000022619	7.15277E-07	0.232824	0.013695529
Linear absolute	17.535026	6.59703E-05	2.08616E-06	0.535026	0.031472118
Exponential linear	17.201953	1.79334E-05	5.67104E-07	0.201953	0.011879588
Linear proportional	17.092519	2.06676E-05	6.53569E-07	0.092519	0.005442294
Jump stochastic	17.055031	0.000020751	6.56204E-07	0.055031	0.003237118
			$\Pi = 100$		
Absolute	17.380953	1.94505E-05	1.94505E-06	0.380953	0.022409000
Square root	17.394206	2.01786E-05	2.01786E-06	0.394206	0.023188588
Proportional	17.402145	2.06651E-05	2.06651E-06	0.402145	0.023655588
Linear absolute	18.023886	6.08044E-05	6.08044E-06	1.023886	0.060228588
Exponential linear	17.325922	0.000016835	1.68350E-06	0.325922	0.019171882
Linear proportional	17.030327	0.00002174	2.17400E-06	0.030327	0.001783941
Jump stochastic	16.974093	2.27266E-05	2.27266E-06	-0.025907	-0.001523941

We can observe that, not only when we use a high number of Monte Carlo simulations Π, but also when we simulate only 100 paths, the standard error of the model is significant as a minimum at the seventh decimal place, meaning that every model is accurate in forecasting the basis spread. The linear absolute volatility model provides the worst results in terms of accuracy, whereas the exponential proportional volatility model fits better than all the other models if we apply 100,000 Monte Carlo simulations.

The same forecasting procedure is implemented through the pricing of a 3/6 Basis Swap, receiving 3-month Libor plus the Basis spread and paying 6-month Libor with one year maturity. We obtain the basis spread Z putting the following formula equal to zero:

$$\begin{aligned} P_{Basis}(0,4\alpha,Z) = \\ E_{\widetilde{\mathbb{P}}}\Big[D(0,\alpha)\alpha(L(0,\alpha)+Z)+D(0,2\alpha)\alpha(\underline{L(\alpha,2\alpha)}+Z)+ \\ D(0,3\alpha)\alpha(\underline{L(2\alpha,3\alpha)}+Z)+D(0,4\alpha)\alpha(\underline{L(3\alpha,4\alpha)}+Z)\Big]- \\ E_{\widetilde{\mathbb{P}}}[D(0,2\alpha)2\alpha L(0,2\alpha)+D(0,4\alpha)2\alpha\underline{L(2\alpha,4\alpha)}]. \end{aligned}$$

The underlined terms are not known and cannot be computed by using market data at inception, on the contrary, they are estimated by applying the Monte Carlo simulation technique.

Table 9 displays results referred to the 3/6 basis swap, and conclusions similar to the case of the 6/12 basis swap apply. We find that the exponential proportional volatility model provides the best fit in terms of accuracy in all cases, that is whatever number of simulations we adopt.

Table 9

3/6 Basis Swap
Summary forecast procedure results and error measures.

This table reports, for seven different volatility functions: *(i)* results of estimated basis spreads Z (measured in basis points), standard deviations σ and standard errors $\hat{\sigma}$, according to different number Π of Monte Carlo simulations; *(ii)* measures of model forecast errors (measured in basis points), obtained as the deviation from the market price, *Abs Err*, and relative deviation from the market price, *Rel Err*.

				$Z_{market}^{5/17/2010} = 20.5\,bps$	
Volatility function	Z	σ	$\hat{\sigma}$	*Abs Err*	*Rel Err*
			$\Pi = 100,000$		
Absolute	20.61616	1.02788E-05	3.25045E-08	0.11616	0.005666341
Square root	20.613363	1.06767E-07	3.37629E-08	0.113363	0.005529902
Proportional	20.611546	1.08916E-05	3.44423E-08	0.111546	0.005441268
Linear absolute	20.315743	0.00003784	1.1966E-07	-0.184257	-0.008988146
Exponential linear	20.625309	8.78283E-06	2.77737E-08	0.125309	0.006112634
Linear proportional	20.641424	1.25009E-05	3.95315E-08	0.141424	0.006898732
Jump stochastic	20.618058	1.03041E-05	3.25847E-08	0.118058	0.005758927
			$\Pi = 10,000$		
Absolute	20.62165	0.000103237	1.03237E-07	0.12165	0.005934146
Square root	20.618611	1.07266E-05	1.07266E-07	0.118611	0.005785902
Proportional	20.616853	1.09458E-05	1.09458E-07	0.116853	0.005700146
Linear absolute	20.333417	3.84132E-05	3.84132E-07	-0.166583	-0.008126000
Exponential linear	20.631179	8.80837E-06	8.80837E-08	0.131179	0.006398976
Linear proportional	20.645391	1.25252E-05	1.25252E-07	0.145391	0.007092244
Jump stochastic	20.607691	1.02608E-05	1.02608E-07	0.107691	0.00525322
			$\Pi = 1,000$		
Absolute	20.645211	1.00267E-05	3.17073E-07	0.145211	0.007083463
Square root	20.643146	1.04297E-05	3.29817E-07	0.143146	0.006982732
Proportional	20.641142	1.06617E-05	3.37153E-07	0.141142	0.006884976
Linear absolute	20.24487	3.92354E-05	1.24073E-06	-0.25513	-0.012445366
Exponential linear	20.65127	8.50343E-06	2.68902E-07	0.15127	0.007379024
Linear proportional	20.683447	1.24778E-05	3.94584E-07	0.183447	0.008948634
Jump stochastic	20.537146	1.03127E-05	3.26116E-07	0.037146	0.001812000
			$\Pi = 100$		
Absolute	20.846529	9.22942E-06	9.22942E-07	0.346529	0.016903854
Square root	20.855551	9.59422E-06	9.59422E-07	0.355551	0.017343951
Proportional	20.860566	9.7753E-06	9.7753E-07	0.360566	0.017588585
Linear absolute	20.747401	3.59379E-05	3.59379E-06	0.247401	0.012068341
Exponential linear	20.808079	8.02658E-06	8.02658E-07	0.308079	0.015028244
Linear proportional	20.632334	1.28819E-05	1.28819E-06	0.132334	0.006455317
Jump stochastic	20.528568	9.81356E-06	9.81356E-07	0.028568	0.001393561

5 Implications of volatility modelling

The model presented in previous sections allows to price a basis swap and determine which one among the seven HJM models evaluates the basis swap in the most accurate way. Clearly, pricing biases can be attributed to diverse factors, from the adopted parameter estimation technique to financial process assumptions, and they measure a sort of model risk. In this Section, we attempt to understand the implications of credit spread volatility function choice on the accuracy of basis swap pricing. Volatility modelling affects interest rate term structure dynamics and indirectly the basis swap price.

In Figures 1 and 2 we plot the absolute forecast error as a function of the Monte Carlo simulation number, both for the case of a 6/12 basis spread and a 3/6 basis spread. The use of the linear absolute volatility function leads to the most misrepresented estimates, indeed pricing errors are the greatest, reaching more than one basis point when we implement 100 simulations. The jump stochastic volatility and the linear proportional one allow to reach a high level of accuracy already by assuming a small number of simulations, and thus they would be convenient if we want to reduce the computational efforts and the time-consuming level. Overall, we find that the model is very stable if we assume a number of simulations greater than 10,000, because in Figures 1 and 2 we show that all curves converge to the average error value of about 0.14 bps, and the choice of a specific volatility function becomes irrelevant in order to obtain the maximal estimate accuracy. On the contrary, if we implement a number of simulations lower than 10,000, results can considerably vary, from a minimal error of 0.025907 bps to a maximum of 1.023886 bps for the 6/12 basis swap, and from a minimum of 0.028568 bps to a maximum of 0.360566 bps for the 3/6 basis swap. Furthermore, in most cases the proposed model overestimates the basis spread.

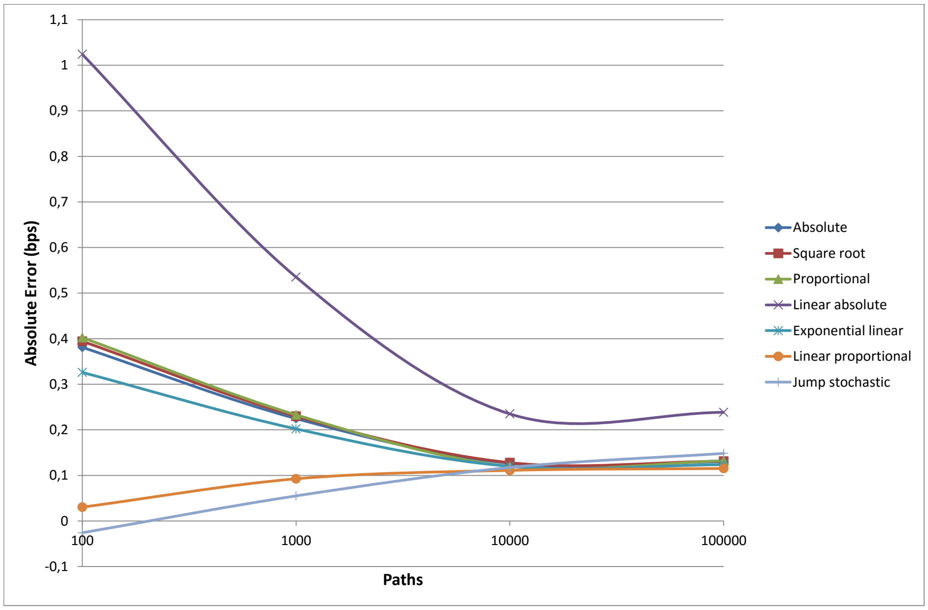

Figure 1: 6/12 Basis swap: forecast error as a function of simulation number

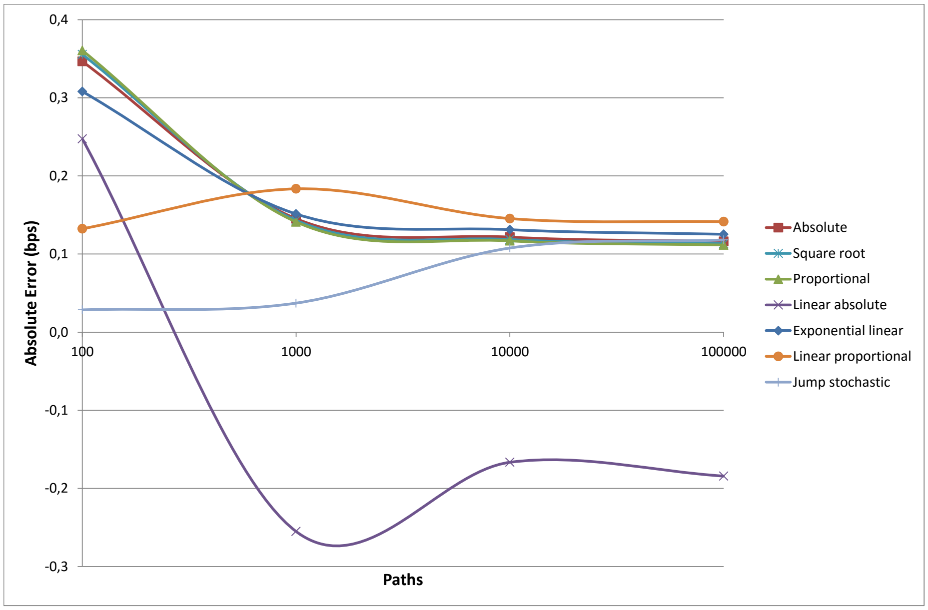

Figure 2: 3/6 Basis swap: forecast error as a function of simulation number

Given this first analysis, we can conclude that an appropriate and reasonable selection of the function for modelling credit spread volatility in derivative pricing provides two advantages:

- increase of estimation accuracy;
- reduction of computational efforts.

In order to study in more detail the nature of the forecast errors and the possible existence of systematic biases in pricing procedure, we carry out an analysis that allows to distinguish between an error due to the simulation technique, in particular to the choice of Π, and an error resulting from the volatility function choice.

Given a chosen number of Monte Carlo simulations and a selected volatility function, we define the computational error θ_{ce} as the relative mispricing per unit of pricing model error as follows:

$$\theta_{ce} = \frac{Rel\,Err}{\hat{\sigma}_{ce}}, \tag{24}$$

where the $Rel\,Err$ is the relative forecast error given by $Rel\,Err = \frac{Z-Z_{market}}{Z_{market}}$, with Z representing the basis spread estimate, whereas $\hat{\sigma}_{ce}$ represents the model residuals standard error multiplied by 10,000. In addition, we define the volatility error θ_{ve} by the following formula that represents the relative mispricing per unit of volatility model error:

$$\theta_{ve} = \frac{Rel\,Err}{\hat{\sigma}_{ve}}, \tag{25}$$

where the $Rel\,Err$ is defined as above and $\hat{\sigma}_{ve}$ represents the residuals standard error of the chosen volatility model.

We are interested in understanding what part of absolute error, given by the deviation of the estimated basis spread from the market data, is explained by the computational error θ_{ce} and what part is affected by the volatility error θ_{ve}.

So we run the following regression, considering all models according to different volatility functions and different Π:

$$Abs\,Err = \beta_0 + \beta_1\theta_{ce} + \beta_2\theta_{ve} + \varepsilon, \tag{26}$$

where $Abs\,Err = Z - Z_{market}$, and ε is the error.

The regression results are shown in Table 10.

Table 10: I Regression Results for equation $Abs\,Err = \beta_0 + \beta_1\theta_{ce} + \beta_2\theta_{ve} + \varepsilon$

	Estimate	Std. Error	t Statistic	P-value
β_0	0.0975441	0.0252009	3.8707	0.0003
β_1	−0.00104832	0.00299109	−0.3505	0.7274
β_2	0.00365436	0.000530430	6.8894	0.0000

Mean	0.170011	Std Deviation	0.179651
Squared residuals sum	0.934701	Std Error	0.132800
R^2	0.473434	Adjusted R^2	0.453564
F statistic	23.82609	P-value(F)	4.15e–08

As we can see from Table 10, coefficient β_2 is not significant, even though the F-test suggests that the two coefficients β_1 and β_2, considered simultaneously, are significant. Furthermore, the standard error of residuals is enough high to believe that there exist some data that are outliers and have to be keep out in the regression analysis. In order to verify this thesis, we plot the scatter plot of the relation between the estimated absolute forecast errors and the effective ones in Figure 3. Given the significant deviation of some points from the dashed line, we find out the existence of outliers and, in order to individuate data that can falsify our analysis, we represent in Figures 4 and 5 the absolute error as a function of the model standard error, according to the different volatility functions. We state that regression input data obtained by the application of the linear absolute volatility model twists results for both 6/12 and 3/6 basis swaps, because they represent anomalous values of model standard errors corresponding to normal levels of absolute forecast error, according to different numbers of simulations Π.

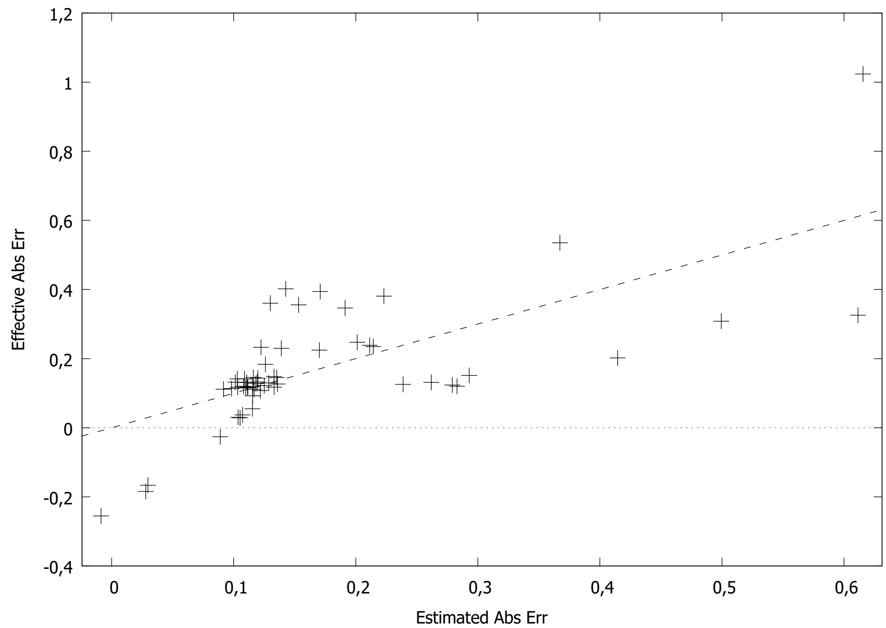

Figure 3: Regression results: Estimated absolute errors vs effective absolute errors

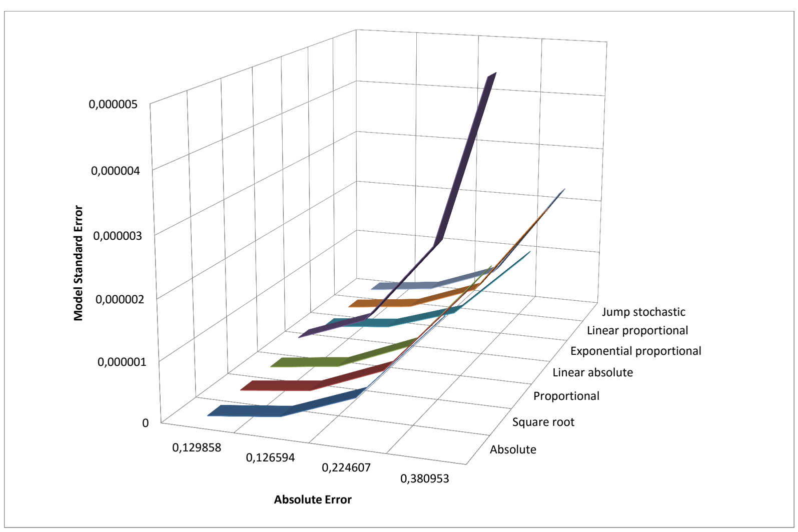

Figure 4: 6/12 Basis swap: Absolute error vs. Model standard error

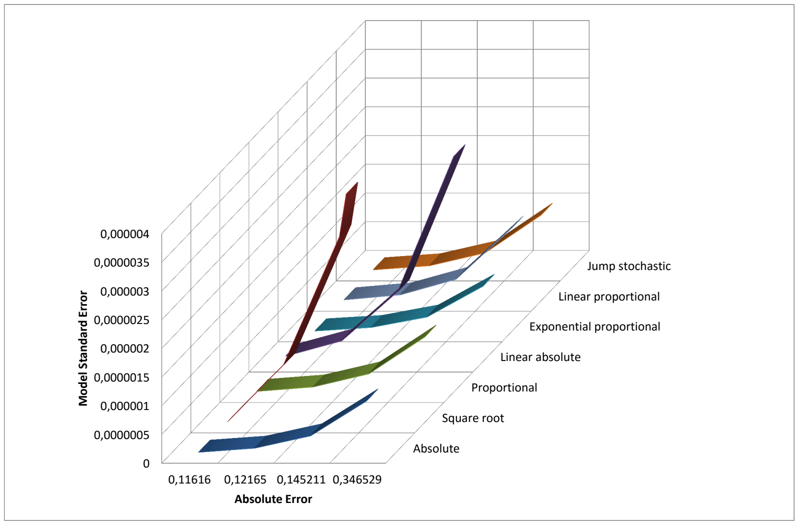

Figure 5: 3/6 Basis swap: Absolute error vs. Model standard error

Consequently, we keep out data referring to the implementation of the linear absolute volatility model from our database, and repeat the regression analysis. The results are summarized in Table 11.

Table 11: II Regression Results for equation $Abs\,Err = \beta_0 + \beta_1\theta_{ce} + \beta_2\theta_{ve} + \varepsilon$

	Estimate	Std. Error	t statistic	p-value
β_0	0.170563	0.0196497	8.6802	0.0000
β_1	0.00124771	0.000447670	2.7871	0.0078
β_2	-0.00556434	0.00220062	-2.5285	0.0151

Mean	0.166349	Std Deviation	0.099749
Squared residuals sum	0.341117	Std Error	0.088049
R^2	0.254702	Adjusted R^2	0.220825
F statistic	7.518409	P-value(F)	0.001553

Both coefficients β_1 and β_2 are significant, and the F statistic of the joint test is rejected. The β_1 estimate is positive, meaning that if the computational error increases of one unit, the chosen model overprices the basis swap by 0.00124771 bps, in fact the absolute error in basis swap pricing is positive. Conversely, the estimate of β_2 is negative, meaning that by increasing the volatility error of one unit, the basis spread is underestimated of 0.00556434 bps.

We now want to understand and quantify the direct relation existing between the relative forecast error and both the pricing model error $\hat{\sigma}_{ce}$ and the volatility model error $\hat{\sigma}_{ve}$. Then, we run the following nonlinear regression analysis:

$$Rel\,Err = \alpha_0 + \alpha_1 \hat{\sigma}_{ce} + \alpha_2 \hat{\sigma}_{ve}^2. \tag{27}$$

The results are shown in Table 12.

Table 12: Nonlinear regression Results for equation $Rel\,Err = \alpha_0 + \alpha_1 \hat{\sigma}_{ce} + \alpha_2 \hat{\sigma}_{ve}^2$

	Estimate	Std. Error	t statistic	p-value
α_0	0.00751133	0.00156269	4.8067	0.0000
α_1	0.142297	0.0531630	2.6766	0.0103
α_2	3.83951	0.724830	5.2971	0.0000

Mean	0,008821	Std Deviation	0,005645
Squared residuals sum	0,001298	Std Error	0,005370
R^2	0,133687	Adjusted R^2	0,095184

Equation (27) approximates the real relative forecast error in an accurate way, in fact the regression standard error is small, significant at the third decimal place.

We assume a linear relation between the relative forecast error and the pricing model error $\hat{\sigma}_{ce}$, and a quadratic relation between the relative forecast error and the volatility model error $\hat{\sigma}_{ve}$. This choice is linked to the fact that the volatility model error is interpreted as a correction of the pricing model error. We find out that, if the model error increases of 1.0E-10, the relative error increases of $\alpha_1 = 0,142297$ bps. On the contrary, the quadratic relation between the relative error and the volatility model error is set in order to individuate a direct effect of the volatility function choice on the pricing model output. However, this effect is smoothed by squaring a quantity smaller than one, that is $\hat{\sigma}_{ve}$. Indeed, although the coefficient α_2, that measures the influence of $\hat{\sigma}_{ve}$ on $Rel\,Err$, is much greater than α_1, the total effect of a variation of $\hat{\sigma}_{ve}$ on the relative error is softened by the nonlinearity of the regression function. In conclusion, our analysis suggests that the choice of an appropriate volatility function allows to increase the accuracy of estimates regardless of the simulation precision.

6 Conclusions

In this paper, we have investigated the credit spread volatility and we have tested seven specific term structure models for credit spreads in the HJM class, that use seven implicit volatility functions as inputs. We have aimed at finding the best representation of the risky Libor rate in interest rate markets after the credit crunch of 2007. From market data we have obtained the daily term structure of forward credit spreads, defined through the implied default intensity of the contributing banks of the Libor corresponding to a chosen tenor. Furthermore, we have evaluated implicit credit spread volatilities. For each model, we have estimated the volatility function parameters. In order to assess the accuracy of models in representing the behaviour of the credit spread, we have tested their ability to predict the price of an interest rate derivative, the basis swap. We have documented systematic discrepancies between the various models and market prices, as functions of the accuracy of the chosen volatility model and the accuracy of the Monte Carlo method. Finally, we have identified and have quantified the implications of the implicit volatility modelling on the accuracy of the basis spread pricing model by defining appropriate pricing error measures and carrying out statistical analyses. Based on these analyses, we have established two criteria to select pricing models for Libor derivatives: *i)* the accuracy of the credit spread volatility model, used for simulating instantaneous defaultable forward rate curve, *ii)* the number of Monte Carlo simulations chosen for the numerical implementation of the pricing model. We have concluded that more importance should be given to the choice of the most accurate volatility model because in this way computational efforts (in terms of number of simulations in the Monte Carlo approach and time-consuming level) could be reduced, even maintaining a high level of precision in price estimates.

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