Even-dimensional slant submanifolds of a $C_5 \oplus C_{12}$-manifold

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Abstract

We study isometric immersions into an almost contact metric manifold which falls in the Chinea-Gonzalez class $C_5 \oplus C_{12}$, under the hypothesis that the Reeb vector field of the ambient space is normal to the considered submanifolds. Particular attention to the case of a slant immersion is paid. We relate immersions into a Kähler manifold to suitable submanifolds of a $C_5 \oplus C_{12}$-manifold. More generally, in the framework of Gray-Hervella, we specify the type of the almost Hermitian structure induced on a non anti-invariant slant submanifold. The cases of totally umbilical or austere submanifolds are discussed.

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Key words: almost contact metric manifold, double-twisted product manifold, slant immersion, totally umbilical submanifold, austere submanifold.

1 Introduction

The theory of slant submanifolds, started by B. Y. Chen in 1990 in the context of almost Hermitian Geometry, has been extended to immersions into a Riemannian manifold endowed with an additional structure ([11, 3, 14, 15, 12]). In particular, in 1996 A. Lotta introduced the concept of a slant submanifold of an almost contact metric (a.c.m.) manifold, showing that there are two types of non anti-invariant slant submanifolds, depending on the position of the Reeb vector field ξ of the ambient space. More precisely, given a non anti-invariant slant submanifold N of an a.c.m. manifold M, with $\dim N = n$, one has: n is even (resp. n is odd) if and only if ξ is normal (resp. ξ is tangent) to N. If n is even, then N inherits from M an almost Hermitian (a.H.) structure.

Slant immersions have been intensively studied when the dimension of the submanifolds is odd and the structure of the ambient space is of a particular type, namely it is cosymplectic, or Sasakian, or α-Kenmotsu ([10], [2], [7] and References therein).

As for as we know, up to now a detailed study of even-dimensional slant submanifolds has not been developed. Note that the condition on the dimension of the submanifold implies a restriction on the a.c.m. structure of the ambient space. In fact, any submanifold N of a contact metric manifold such that ξ is normal to N is anti-invariant ([11]). This makes meaningful the investigation of even-dimensional slant submanifolds only when the a.c.m. structure of the ambient space in not contact.

In this work, we relate immersions in a.H. manifolds to submanifolds of suitable a.c.m. manifolds. Firstly, starting by a slant submanifold $(\widehat{N}, \widehat{f})$ of an a.H. manifold $\widehat{M}$, for any smooth positive function $\lambda\colon \widehat{M} \to \mathbb{R}$ and any open interval $I \subset \mathbb{R}$, we consider a particular slant immersion $f_\lambda\colon \widehat{N} \to I \times \widehat{M}$, with the same slant angle as $\widehat{f}$. The manifold $I \times \widehat{M}$ is endowed with an a.c.m. structure naturally associated with the a.H. structure on $\widehat{M}$, the Reeb vector field is orthogonal to $\widehat{N}$ and, if $\widehat{M}$ is a Kähler manifold, then $I \times \widehat{M}$ falls in the Chinea-Gonzalez class C_{12}. We also explain a method to construct a family of slant immersions into a C_5-manifold starting by a pair $(\widehat{f}, \lambda)$, $\widehat{f}$ being a slant immersion into a Kähler manifold and λ a smooth positive function.

More generally, we study slant immersions into a $C_5 \oplus C_{12}$-manifold such that the Reeb vector field is normal to the considered submanifolds. In particular, we prove that the a.H. structure induced on a slant submanifold is almost Kähler and state a condition on the behavior of the Weingarten operators. This allows us to prove that the a.H. structure on the considered submanifold is of Kähler type, under additional hypotheses. The cases of totally umbilical or austere submanifolds are examined in detail, also.

In this work all manifolds are assumed to be connected.

2 Preliminaries

Given an a.H. manifold $(\widehat{M}, \widehat{J}, \widehat{g})$, let $\widehat{f}\colon (\widehat{N}, \widehat{g}') \to (\widehat{M}, \widehat{g})$ be an isometric immersion. For any $x \in \widehat{N}$, $X \in T_x\widehat{N}$, we adopt the identifications $x \equiv \widehat{f}(x)$ and $X \equiv (\widehat{f}_*)_x X$, $(\widehat{f}_*)_x$ being the tangential map. For any $X \in T\widehat{N}$, one puts $\widehat{J}X = \widehat{P}X + \widehat{F}X$, where $\widehat{P}X$ and $\widehat{F}X$ denote the tangential and normal components of $\widehat{J}X$, respectively. Analogously, for any $V \in T^\perp\widehat{N}$, we put $\widehat{J}V = \widehat{t}V + \widehat{n}V$, $\widehat{t}V$, $\widehat{n}V$ being the tangential and normal components of $\widehat{J}V$. So, one defines smooth maps $\widehat{P}\colon T\widehat{N} \to T\widehat{N}$, $\widehat{F}\colon T\widehat{N} \to T^\perp\widehat{N}$, $\widehat{t}\colon T^\perp\widehat{N} \to T\widehat{N}$, $\widehat{n}\colon T^\perp\widehat{N} \to T^\perp\widehat{N}$, that induce linear maps on each fibre. For any $X, Y \in T\widehat{N}$ one has $\widehat{g}'(\widehat{P}X, Y) = -\widehat{g}'(X, \widehat{P}Y)$, so that at any point $x \in \widehat{N}$ the operator $\widehat{Q} = \widehat{P}^2$ is a self-adjoint endomorphism of $T_x\widehat{N}$, whose non-zero eigenvalues belong to $[-1, 0[$ and have even multiplicity. We denote by the same symbol the tensor fields on $\widehat{N}$ determined by $\widehat{P}$, $\widehat{Q}$.

For any $x \in \widehat{N}$, $X \in T_x\widehat{N}$, $X \neq 0$, the angle $\theta(X) \in [0, \frac{\pi}{2}]$ between $\widehat{J}X$ and $T_x\widehat{N}$ is called the Wirtinger angle of X. The immersion $\widehat{f}$ is called a slant immersion if the angle $\theta(X)$ is a constant θ, namely it is independent of the choice of $(x, X) \in T\widehat{N}$. In this case, one says that θ is the slant angle of $\widehat{N}$ in $\widehat{M}$ and writes $sla(\widehat{N}) = \theta$. If $sla(\widehat{N}) = \theta \neq \frac{\pi}{2}$, then the dimension of $\widehat{N}$ is even and $(\frac{1}{\cos\theta}\widehat{P}, \widehat{g}')$ is an a.H. structure on $\widehat{N}$.

Let $(M, \varphi, \xi, \eta, g)$ be an a.c.m. manifold and $f\colon (N, g') \to (M, g)$ an isometric immersion. For any $X \in TN$, one puts $\varphi X = PX + FX$, PX and FX denoting the tangential and normal components of φX. Also, for any $V \in T^\perp N$, we put $\varphi V = tV + nV$, tV and nV being the tangential and normal components of φV. This allows us to define smooth maps $P\colon TN \to TN$, $F\colon TN \to T^\perp N$, $t\colon T^\perp N \to TN$, $n\colon T^\perp N \to T^\perp N$ inducing linear maps on each fibre. In particular, for any $X, Y \in TN$ one has $g'(PX, Y) = -g'(X, PY)$, hence at any point $x \in N$ $Q = P^2$ is a self-adjoint endomorphism of T_xN, its non-zero eigenvalues belong to $[-1, 0[$ and have even multiplicity. Furthermore, if the Reeb vector field ξ is normal to N, the maps P, t, F, n are related by

$$\begin{aligned} &P^2 + tF = -I_{TN}, \quad FP + nF = 0, \\ &Pt + tn = 0, \quad Ft + n^2 = -I_{T^\perp N} + \eta \otimes \xi. \end{aligned} \tag{2.1}$$

As in [11], the immersion f is said to be a slant immersion if for any $x \in N$, $X \in T_xN$, such that X, ξ are linearly independent, the angle $\theta(X) \in [0, \frac{\pi}{2}]$ between φX and T_xN is a constant θ. In this case, we put $\mathrm{sla}(N) = \theta$, θ is named the slant angle of N in M and (N, f) is called a slant submanifold of M. In particular, if $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$), (N, f) is an invariant (resp. anti-invariant) submanifold. If $\mathrm{sla}(N) = \theta \neq 0, \frac{\pi}{2}$, (N, f) is called a proper slant submanifold.

Given an isometric immersion $f\colon (N, g') \to (M, \varphi, \xi, \eta, g)$ such that ξ is normal to N, one has: (N, f) is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that $Q = -\lambda I_{TN}$. Moreover, if $\mathrm{sla}(N) = \theta$, then $\lambda = \cos^2\theta$ and for any $X, Y \in TN$ one gets

$$g'(PX, PY) = (\cos^2\theta) g'(X, Y), \quad g(FX, FY) = (\sin^2\theta) g'(X, Y). \tag{2.2}$$

It follows that, if $\theta \neq \frac{\pi}{2}$, then $(J = \frac{1}{\cos\theta} P, g')$ is an a.H. structure on N, called the a.H. structure induced on N by f ([2], [11]).

In [11], the author links slant submanifolds of an a.H. manifold with slant submanifolds of a suitable a.c.m. manifold. More precisely, given an a.H. manifold $(\widehat{M}, \widehat{J}, \widehat{g})$, we endow the product manifold $\widehat{M} \times \mathbb{R}$ with the a.c.m. structure (φ, ξ, η, g) defined by

$$\begin{aligned} &\varphi(X, a\frac{\partial}{\partial t}) = (\widehat{J}X, 0), \quad \eta(X, a\frac{\partial}{\partial t}) = a, \\ &\xi = (0, \frac{\partial}{\partial t}), \quad g = \widehat{g} + dt \otimes dt, \end{aligned} \tag{2.3}$$

for any $X \in \Gamma(T\widehat{M})$, $a \in \mathfrak{F}(\widehat{M} \times \mathbb{R})$. If $\widehat{f}\colon (\widehat{N}, \widehat{g'}) \to (\widehat{M}, \widehat{g})$ is an isometric immersion, one considers the immersion $f_0\colon \widehat{N} \to \widehat{M} \times \mathbb{R}$ such that $f_0(x) = (\widehat{f}(x), 0)$, for any $x \in \widehat{N}$. Obviously, the vector field ξ is normal to $(\widehat{N}, f_0)$. Moreover, $(\widehat{N}, \widehat{f})$ is slant in $\widehat{M}$ with $\mathrm{sla}(\widehat{N}) = \theta$ if and only if $(\widehat{N}, f_0)$ is slant in $\widehat{M} \times \mathbb{R}$ with $\mathrm{sla}(\widehat{N}) = \theta$.

Now, we focus on a wider class of a.c.m. manifolds strictly related to a.H. manifolds. Given an a.H. manifold $(\widehat{M}, \widehat{J}, \widehat{g})$, an open inteval $I \subset \mathbb{R}$ and two smooth positive functions $\lambda_1, \lambda_2\colon I \times \widehat{M} \to \mathbb{R}$, on the product manifold $I \times \widehat{M}$ one considers the a.c.m. structure $(\varphi, \xi, \eta, g_{(\lambda_1,\lambda_2)})$ given by

$$\begin{aligned} &\varphi(a\frac{\partial}{\partial t}, X) = (0, \widehat{J}X), \quad \eta(a\frac{\partial}{\partial t}, X) = a\lambda_1, \\ &\xi = \frac{1}{\lambda_1}(\frac{\partial}{\partial t}, 0), \quad g_{(\lambda_1,\lambda_2)} = \lambda_1^2 \pi_1^*(dt \otimes dt) + \lambda_2^2 \pi_2^*(\widehat{g}), \end{aligned} \tag{2.4}$$

for any $a \in \mathfrak{F}(I \times \widehat{M}), X \in \Gamma(T\widehat{M})$, $\pi_1\colon I \times \widehat{M} \to I$, $\pi_2\colon I \times \widehat{M} \to \widehat{M}$ denoting the canonical projections. Note that $g_{(\lambda_1,\lambda_2)}$ is the double-twisted product of the Euclidean metric g_0 and $\widehat{g}$ ([13]). The a.c.m. manifold $I \times_{(\lambda_1,\lambda_2)} \widehat{M} = (I \times \widehat{M}, \varphi, \xi, \eta, g_{(\lambda_1,\lambda_2)})$ is named the double-twisted product manifold of (I, g_0) and $(\widehat{M}, \widehat{J}, \widehat{g})$ by (λ_1, λ_2). If $\lambda_1 \equiv 1$, then $I \times_{(1,\lambda_2)} \widehat{M}$, which is denoted by

$I \times_{\lambda_2} \widehat{M}$, is called the twisted product manifold of (I, g_0) and $(\widehat{M}, \widehat{J}, \widehat{g})$ by λ_2 ([8]). If $\lambda_2 \equiv 1$, the manifold $I \times_{(\lambda_1,1)} \widehat{M}$ is denoted by ${}_{\lambda_1}I \times \widehat{M}$. Furthermore, when λ_1 is independent of the coordinate t and λ_2 only depends on t, $g_{(\lambda_1,\lambda_2)}$ is just the double-warped product metric of g_0 and $\widehat{g}$ by (λ_1, λ_2) and $I \times_{(\lambda_1,\lambda_2)} \widehat{M}$ is called a double-warped product manifold, as well as $I \times_{\lambda_2} \widehat{M}$ is said to be a warped product manifold. Obviously, the warped product manifold of $(\mathbb{R}, g_0)$ and $(\widehat{M}, \widehat{J}, \widehat{g})$ by $\lambda_2 \equiv 1$ is identified with the product manifold $\widehat{M} \times \mathbb{R}$ endowed with the structure defined in (2.3).

Applying the theory developed in [6, 8], we are also able to specify the Chinea-Gonzalez class of the just mentioned manifolds. Firstly, in Table 1 we list the defining conditions of any a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ which falls in the Chinea-Gonzalez class $C_{1-5} \oplus C_{12} = \bigoplus_{1 \le i \le 5} C_i \oplus C_{12}$ or in suitable subclasses. We also use the symbols $C_{1-5} = \bigoplus_{1 \le i \le 5} C_i$ and $C_{1-4} \oplus C_{12} = \bigoplus_{1 \le i \le 4} C_i \oplus C_{12}$. Putting $\dim M = 2m + 1$, these conditions are formulated in terms of the covariant derivatives $\nabla\varphi$, $\nabla\eta$, ∇ denoting the Levi-Civita connection of M.

Table 1

Classes	Defining conditions
$C_{1-5} \oplus C_{12}$	$(\nabla_\xi \varphi)Y = -\eta(Y)\varphi(\nabla_\xi \xi) - (\nabla_\xi \eta)\varphi Y \xi$, $(\nabla_X \eta)Y = -\frac{\delta\eta}{2m} g(\varphi X, \varphi Y) + \eta(X)(\nabla_\xi \eta)Y$
C_{1-5}	$\nabla_\xi \varphi = 0, \quad (\nabla_X \eta)Y = -\frac{\delta\eta}{2m} g(\varphi X, \varphi Y)$
$C_{1-4} \oplus C_{12}$	$(\nabla_\xi \varphi)Y = -\eta(Y)\varphi(\nabla_\xi \xi) - (\nabla_\xi \eta)\varphi Y \xi$, $(\nabla_X \eta)Y = \eta(X)(\nabla_\xi \eta)Y$
$C_5 \oplus C_{12}$	$(\nabla_X \varphi)Y = -\frac{\delta\eta}{2m}\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}$ $-\eta(X)\{(\nabla_\xi \eta)\varphi Y \xi + \eta(Y)\varphi(\nabla_\xi \xi)\}$
C_5	$(\nabla_X \varphi)Y = -\frac{\delta\eta}{2m}\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}$
C_{12}	$(\nabla_X \varphi)Y = -\eta(X)\{(\nabla_\xi \eta)\varphi Y \xi + \eta(Y)\varphi(\nabla_\xi \xi)\}$

Considering an a.H. manifold $(\widehat{M}, \widehat{J}, \widehat{g})$, any double-twisted product manifold $I \times_{(\lambda_1,\lambda_2)} \widehat{M}$ belongs to $C_{1-5} \oplus C_{12}$ and any twisted product manifold $I \times_{\lambda_2} \widehat{M}$ falls in C_{1-5}. If λ_1 is independent of the coordinate t, the manifold ${}_{\lambda_1}I \times \widehat{M}$ falls in the class $C_{1-4} \oplus C_{12}$. Furthermore, under suitable restriction on the Gray-Hervella class of $(\widehat{M}, \widehat{J}, \widehat{g})$, $I \times_{(\lambda_1,\lambda_2)} \widehat{M}$ falls in a particular subclass of $C_{1-5} \oplus C_{12}$. In particular, we assume that $(\widehat{J}, \widehat{g})$ is a Kähler structure. Then, if λ_2 is constant on $\widehat{M}$, $I \times_{(\lambda_1,\lambda_2)} \widehat{M}$ is a $C_5 \oplus C_{12}$-manifold and when λ_1 is independent of the Euclidean coordinate, the manifold ${}_{\lambda_1}I \times \widehat{M}$ belongs to C_{12}. Finally, we recall that any warped product manifold $I \times_{\lambda_2} \widehat{M}$ is a C_5-manifold and it is called an α-Kenmotsu manifold, where $\alpha = -\frac{1}{2m}\delta\eta = \xi(\log \lambda_2)$.

3 Some methods to construct slant immersions

In this section, firstly we extend the procedure given in Section 2 that allows us to obtain even-dimensional slant submanifolds of an a.c.m. manifold, starting by a slant immersion into an a.H. manifold.

Let $(\widehat{M}, \widehat{J}, \widehat{g})$ be an a.H. manifold and $\widehat{f}\colon (\widehat{N}, \widehat{g}') \to (\widehat{M}, \widehat{g})$ an isometric immersion. Given an open interval $I \subset \mathbb{R}$, $0 \in I$, and a smooth positive function $\lambda\colon \widehat{M} \to \mathbb{R}$, the map

$$f_\lambda\colon (\widehat{N}, \widehat{g}') \to (I \times \widehat{M}, g_{(\lambda,1)} = \lambda^2 dt \otimes dt + \widehat{g}), \quad f_\lambda(x) = (0, \widehat{f}(x)), \quad x \in \widehat{N} \tag{3.1}$$

is an isometric immersion. Hence $N_\lambda = (\widehat{N}, f_\lambda)$ is a Riemannian submanifold of the a.c.m. manifold ${}_\lambda I \times \widehat{M}$. For any $x \in \widehat{N}$, $X \in T_x\widehat{N}$ one has $((f_\lambda)_*)_x X = (0, (\widehat{f}_*)_x X) \equiv (\widehat{f}_*)_x X$, so the tangent spaces T_xN_λ, $T_x\widehat{N}$ are identified. As for as regard the normal spaces $T_x^\perp N_\lambda$, $T_x^\perp \widehat{N}$, one has $T_x^\perp N_\lambda \cong I \times T_x^\perp \widehat{N}$. It follows that the vector field $\xi_{|\widehat{N}} = \frac{1}{\lambda \circ \widehat{f}}(\frac{\partial}{\partial t}, 0)$ is a section of $T^\perp N_\lambda$, that is ξ is normal to N_λ. To relate the second fundamental forms h, h_λ of $\widehat{N}$, N_λ we recall that the Levi-Civita connection ∇ of ${}_\lambda I \times \widehat{M}$ acts as

$$\begin{aligned} &\nabla_X Y = \widehat{\nabla}_X Y, \quad \nabla_\xi \xi = -\mathrm{grad} \log \lambda, \\ &\nabla_X \xi = 0, \quad \nabla_\xi X = X(\log \lambda)\xi, \end{aligned} \tag{3.2}$$

for any $X, Y \in \Gamma(T\widehat{M})$, where $\widehat{\nabla}$ is the Levi-Civita connection of $(\widehat{M}, \widehat{g})$ and grad is evaluated with respect to the metric $g_{(\lambda,1)}$ ([8, 13]). Applying (3.2) and the Gauss equation, we get

$$h_\lambda(X, Y) = (0, h(X, Y)) \equiv h(X, Y), \quad X, Y \in \Gamma(T\widehat{N}). \tag{3.3}$$

It follows that the Weingarten operators corresponding to the immersions f_λ, $\widehat{f}$ are related by

$$(A_\lambda)_V X = A_{V - \eta(V)\xi} X, \quad x \in \widehat{N}, \quad X \in T_x\widehat{N}, \quad V \in T_x^\perp N_\lambda. \tag{3.4}$$

By (3.3), (3.4) one easly proves the equivalences:

i) $(\widehat{N}, \widehat{f})$ is totally geodesic if and only if N_λ is totally geodesic.

ii) $(\widehat{N}, \widehat{f})$ is totally umbilical (resp. austere) if and only if N_λ is totally umbilical (resp. austere).

For any $X \in T\widehat{N}$, let $P_\lambda X$ (resp. $F_\lambda X$) be the tangential (resp. normal) component of φX. Since $\varphi X \equiv \widehat{J}X = \widehat{P}X + \widehat{F}X$, we get

$$P_\lambda X = \widehat{P}X, \quad F_\lambda X = \widehat{F}X. \tag{3.5}$$

Proposition 3.1. *Let $(\widehat{M}, \widehat{J}, \widehat{g})$ be an a.H. manifold, $(\widehat{N}, \widehat{f})$ a submanifold of $\widehat{M}$, $I \subset \mathbb{R}$ an open interval, $0 \in I$, and $\lambda\colon \widehat{M} \to \mathbb{R}$ a smooth positive function. Then, $(\widehat{N}, \widehat{f})$ is a slant submanifold with sla$(\widehat{N}) = \theta$ if and only if N_λ is a slant submanifold with sla$(N_\lambda) = \theta$.*

Proof. Given $x \in \widehat{N}$, $X \in T_x\widehat{N}$, $X \neq 0$, we denote by $\theta_\lambda(X)$ the angle between φX and T_xN_λ. By (2.4), (3.5), we have

$$\cos\theta_\lambda(X) = \frac{g_{(\lambda,1)}(\varphi X, P_\lambda X)}{||\varphi X||_\lambda ||P_\lambda X||_\lambda} = \frac{||\widehat{P}X||}{||X||} = \cos\theta(X),$$

where $||\cdot||_\lambda$ denotes the norm induced by $g_{(\lambda,1)}$ and $\theta(X)$ is the Wirtinger angle of X. $\square$

Since the a.c.m. manifold ${}_\lambda I \times \widehat{M}$ falls in the Chinea-Gonzalez class $C_{1-4} \oplus C_{12}$, by Proposition 3.1 we get a method to construct a family of slant submanifolds of $C_{1-4} \oplus C_{12}$-manifolds, starting by a slant submanifold of an a.H. manifold. In particular, if $I \subset \mathbb{R}$ is an open interval and $0 \in I$, a slant immersion $\widehat{f}\colon \widehat{N} \to \widehat{M}$ into a Kähler manifold gives rise to the family of slant immersions $\{f_\lambda; \lambda \in C^\infty(\widehat{M}), \lambda > 0\}$ into the manifolds ${}_\lambda I \times \widehat{M}$, $\lambda \in C^\infty(\widehat{M})$, $\lambda > 0$, which belong to the class C_{12}.

We remark that, considering a slant submanifold $(\widehat{N}, \widehat{f})$ of a Kähler manifold $\widehat{M}$ with $\mathrm{sla}(\widehat{N}) = \theta \neq \frac{\pi}{2}$, $(\frac{1}{\cos\theta}\widehat{P}, \widehat{g}')$ is an almost Kähler structure and it coincides with the a.H. structure $(\frac{1}{\cos\theta}P_\lambda, \widehat{g}')$ induced on $\widehat{N}$ by f_λ. In particular, if $\dim \widehat{N} = 2$, $(\widehat{N}, \frac{1}{\cos\theta}\widehat{P}, \widehat{g}')$ is a Kähler manifold.

Many explicit examples can be obtained starting by the main examples of slant immersions given in [4], [5]. In next Examples 3.1, 3.3 we provide families of austere, but non-totally geodesic, slant immersions as well as, for any $\alpha \in [0, \frac{\pi}{2}]$, in Example 3.2 we obtain a family of totally geodesic slant immersions with slant angle α.

Example 3.1. The map $\widehat{f}\colon \mathbb{R}^2 \to \mathbb{R}^4$ such that

$$\widehat{f}(x^1, x^2) = (x^1, x^2, e^{x^1}\cos x^2, e^{x^1}\sin x^2)$$

defines a holomorphic immersion into the Kähler manifold $(\mathbb{R}^4, \widehat{J}, g_0)$, g_0 being the Euclidean metric and $\widehat{J}$ acting as

$$\widehat{J}(y^1, y^2, y^3, y^4) = (-y^2, y^1, -y^4, y^3).$$

Moreover $(\mathbb{R}^2, \widehat{f})$ is austere, but non-totally geodesic. So, given an open interval $I \subset \mathbb{R}$ with $0 \in I$, for any $\lambda \in C^\infty(\mathbb{R}^4)$, $\lambda > 0$, the map $f_\lambda\colon \mathbb{R}^2 \to {}_\lambda I \times \mathbb{R}^4$ acting as $f_\lambda(x^1, x^2) = (0, \widehat{f}(x^1, x^2))$ is an austere invariant immersion into a C_{12}-manifold.

Example 3.2 ([4])**.** For any $\alpha \in [0, \frac{\pi}{2}]$ the map $\widehat{f}^\alpha\colon \mathbb{R}^2 \to \mathbb{R}^4$ acting as

$$\widehat{f}^\alpha(x^1, x^2) = (x^1 \sin\alpha, x^2, 0, x^1\cos\alpha)$$

is an isometric totally geodesic immersion with respect to the Euclidean metrics g_0' and g_0 on $\mathbb{R}^2$, $\mathbb{R}^4$, respectively. Let J_0 be the canonical almost complex structure on $\mathbb{R}^4$ and $X_i = (\widehat{f}^\alpha)_*(\frac{\partial}{\partial x^i})$, $i \in \{1, 2\}$. By direct calculus the tangential components of $J_0(X_1)$, $J_0(X_2)$ are given by

$$\widehat{P}X_1 = -\cos\alpha X_2, \quad \widehat{P}X_2 = \cos\alpha X_1.$$

This implies that $\widehat{P} = -\cos\alpha J_0$ and $(\mathbb{R}^2, \widehat{f}^\alpha)$ is a slant submanifold of the Kähler manifold $(\mathbb{R}^4, J_0, g_0)$ with slant angle α. It follows that, for any pair (I, λ), I being an open interval, with $0 \in I$, and $\lambda \in C^\infty(\mathbb{R}^4)$, $\lambda > 0$, the map $f_\lambda^\alpha\colon \mathbb{R}^2 \to {}_\lambda I \times \mathbb{R}^4$ such that $f_\lambda^\alpha(x^1, x^2) = (0, \widehat{f}^\alpha(x^1, x^2))$ is a totally geodesic slant immersion with slant angle α.

Example 3.3. Let $f = (\widehat{f}, \widehat{f}^0)\colon \mathbb{R}^4 \to \mathbb{R}^8$ be the Riemannian product immersion of the immersions $\widehat{f}$ given in Example 3.1 and $\widehat{f}^\alpha$, $\alpha = 0$, occurring in Example 3.2. So, f acts as

$$f(x^1, x^2, x^3, x^4) = (x^1, x^2, e^{x^1}\cos x^2, e^{x^1}\sin x^2, 0, x^4, 0, x^3).$$

On $\mathbb{R}^8$ we consider the a.H. structure (J, g_0), g_0 being the Euclidean metric and $J = (\widehat{J}, J_0)$ acting as

$$J(y^1, y^2, y^3, y^4, y^5, y^6, y^7, y^8) = (-y^2, y^1, -y^4, y^3, -y^7, -y^8, y^5, y^6).$$

One proves that $(\mathbb{R}^8, J, g_0)$ is a Kähler manifold and f is a holomorphic non-totally geodesic austere immersion. Let $I \subset \mathbb{R}$ be an open interval, with $0 \in I$. For any smooth positive function $\lambda\colon \mathbb{R}^8 \to \mathbb{R}$ the manifold ${}_\lambda I \times \mathbb{R}^8$ is in the class C_{12} and the map $f_\lambda\colon \mathbb{R}^4 \to {}_\lambda I \times \mathbb{R}^8$, $f_\lambda(x) = (0, f(x))$, is an invariant austere immersion.

Now, we explain a method to obtain examples of slant immersions into a C_5-manifold.

We recall that, if $(M, \varphi, \xi, \eta, g)$ is a manifold in the class C_5, $\dim M = 2m+1$, putting $\alpha = -\frac{\delta\eta}{2m}$, one has $\nabla_X \xi = \alpha\{X - \eta(X)\xi\}$. It follows that the distribution D associated with the subbundle $\mathrm{Ker}\eta$ of TM is integrable and defines an umbilical foliation. If $(\overline{M}, i)$ is a leaf of D, $i\colon \overline{M} \to M$ being the inclusion map, its second fundamental form acts as $h(X,Y) = -\alpha_{|\overline{M}}\, g(X,Y)\xi$ and, if $m \geq 2$, $\alpha_{|\overline{M}}$ is constant. Moreover, for any m, $(J = \varphi_{|T\overline{M}}, i^*g)$ is a Kähler structure.

Theorem 3.1. *Let $(M, \varphi, \xi, \eta, g)$ be an α-Kenmotsu manifold and $(\overline{M}, i)$ a leaf of the distribution D. Assume that $\overline{N} = (N, \overline{f})$ is a submanifold of $\overline{M}$ and put $\widetilde{f} = i \circ \overline{f}$, $\widetilde{N} = (N, \widetilde{f})$. Then, the following properties hold*

i) The second fundamental forms $\widetilde{h}$, $\overline{h}$ of $\widetilde{N}$, $\overline{N}$ are related by

$$\widetilde{h}(X,Y) = \overline{h}(X,Y) - (\alpha \circ \widetilde{f})g(X,Y)\xi.$$

ii) $\overline{N}$ is a slant submanifold with $sla(\overline{N}) = \theta$ if and only if $\widetilde{N}$ is a slant submanifold and $sla(\widetilde{N}) = \theta$. Moreover, if $sla(\widetilde{N}) = \theta \neq \frac{\pi}{2}$, then $\widetilde{N}$ inherits from M an almost Kähler structure.

Proof. Firstly, we observe that $TM_{|\overline{M}} = D \oplus <\xi>$ and $T^\perp\widetilde{N} = T^\perp\overline{N} \oplus <\xi>$. Since $h(X,Y) = -\alpha_{|\overline{M}}\, g(X,Y)\xi$, *i)* directly follows by the Gauss equation.

For any $X \in TN$, since $JX \equiv \varphi X$, one gets $\overline{P}X \equiv \widetilde{P}X$, $\overline{P}X$ (resp. $\widetilde{P}X$) denoting the tangential component of JX (resp. φX) with respect to $\overline{N}$ (resp. $\widetilde{N}$). This entails the equivalence in *ii)*. Finally, since $(\overline{M}, J, i^*g)$ is a Kähler manifold, if $\mathrm{sla}(\overline{N}) = \theta \neq \frac{\pi}{2}$, the a.H. structure $(\frac{1}{\cos\theta}\overline{P}, \overline{g} = \overline{f}^*(i^*g)) = (\frac{1}{\cos\theta}\widetilde{P}, \widetilde{f}^*g)$ is almost Kähler. □

Let $(\widehat{M}, \widehat{J}, \widehat{g})$ be a Kähler manifold, $I \subset \mathbb{R}$ an open interval and $\lambda\colon I \to \mathbb{R}$ a smooth positive function. As remarked in Section 2, the warped product manifold $I \times_\lambda \widehat{M}$ is an α-Kenmotsu manifold, $\alpha = \xi(\log\lambda) = \frac{\lambda'}{\lambda}$. Using the canonical identification $X \equiv (0, X)$, for any $X \in T\widehat{M}$, we get that the distribution D on $I \times_\lambda \widehat{M}$ is identified with $T\widehat{M}$. Thus, given a point $(t_0, x_0) \in I \times \widehat{M}$, the leaf of D through (t_0, x_0) is the submanifold $(\widehat{M}, f_{t_0})$, $f_{t_0}\colon \widehat{M} \to I \times \widehat{M}$ acting as $f_{t_0}(x) = (t_0, x)$, and the α-Kenmotsu structure on $I \times_\lambda \widehat{M}$ induces on $(\widehat{M}, f_{t_0})$

the Kähler structure $(\widehat{J}, \lambda(t_0)^2\widehat{g})$. By Theorem 3.1, considering a slant submanifold $(N, \overline{f})$ of $\widehat{M}$, with $\mathrm{sla}(N) = \theta$, we get that $\widetilde{f} = f_{t_0} \circ \overline{f} \colon N \to I \times_\lambda \widehat{M}$ is slant with angle θ.

The next example clarifies this procedure.

Example 3.4. Given $k \in \mathbb{R}$, $k > 0$, let $\overline{f} \colon \mathbb{R}^2 \to \mathbb{R}^4$ be the map acting as

$$\begin{aligned}\overline{f}(x^1, x^2) = (&e^{kx^1} \cos x^1 \cos x^2, e^{kx^1} \sin x^1 \cos x^2, e^{kx^1} \cos x^1 \sin x^2,\\ &e^{kx^1} \sin x^1 \sin x^2).\end{aligned}$$

It is easy to see that $\overline{f}$ is a slant immersion into the Kähler manifold $(\mathbb{R}^4, J_0, g_0)$, (J_0, g_0) being the canonical Hermitian structure and $\mathrm{sla}(\mathbb{R}^2) = \arccos(\frac{k}{\sqrt{k^2+1}})$. On $\mathbb{R}^2$ we consider the metric $\widehat{g'}$ represented, with respect to the natural frame, by the matrix

$$\begin{pmatrix} (k^2+1)e^{2kx^1} & 0 \\ 0 & e^{2kx^1} \end{pmatrix}.$$

Given an open inteval I , let $\lambda \colon I \to \mathbb{R}$ be a smooth function, $\lambda > 0$, and (t_0, x_0) a point of $I \times \mathbb{R}^4$. The leaf of the distribution $D =< \xi >^\perp$ on the warped product manifold $I \times_\lambda \mathbb{R}^4$ through (t_0, x_0) is identified with $(\mathbb{R}^4, J_0, \lambda(t_0)^2 g_0)$. By Theorem 3.1 the isometric immersion $\widetilde{f} \colon (\mathbb{R}^2, J_0, \lambda(t_0)^2\widehat{g'}) \to I \times_\lambda \mathbb{R}^4$ acting as $\widetilde{f}(x^1, x^2) = (t_0, \overline{f}(x^1, x^2))$ defines a slant submanifold with angle $\theta = \arccos(\frac{k}{\sqrt{k^2+1}})$.

4 Submanifolds of a $C_5 \oplus C_{12}$-manifold

The content of Section 3 motivates the study of slant immersions into $C_5 \oplus C_{12}$-manifolds. We recall some properties of these manifolds, that are locally described in [8].

Let $(M, \varphi, \xi, \eta, g)$ be an a.c.m. manifold in the class $C_5 \oplus C_{12}$ and ∇ its Levi-Civita connection. We put $\dim M = 2m + 1$, $\alpha = -\frac{\delta\eta}{2m}$. The function α determines the C_5-component of $\nabla\varphi$, as well as $\nabla_\xi\xi$ defines its C_{12}-component. In fact, for any $X, Y \in \Gamma(TM)$ one has

$$\begin{aligned}(\nabla_X\varphi)Y =&\alpha\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\}\\ &- \eta(X)\{\eta(Y)\varphi(\nabla_\xi\xi) + g(\nabla_\xi\xi, \varphi Y)\xi\}.\end{aligned} \tag{4.1}$$

It follows $d\eta = \eta\wedge\nabla_\xi\eta$, so the distribution D associated with the subbundle $\mathrm{Ker}\eta$ of TM is integrable. Generally, if $\nabla_\xi\xi \neq 0$ and $\dim M \geq 5$, α is not constant on the leaves of D. In fact, being $d\Phi = 2\alpha\eta \wedge \Phi$, where Φ is the fundamental 2-form of M, one gets $(d\alpha - \alpha\nabla_\xi\eta) \wedge \eta = 0$ and for any X orthogonal to ξ we have $X(\alpha) = \alpha g(\nabla_\xi\xi, X)$.

Now, we consider an isometric immersion $f \colon (N, g') \to (M, \varphi, \xi, \eta, g)$ such that ξ is normal to N. Let ∇' (resp. $\nabla^\perp$) be the Levi-Civita (resp. normal) connection of (N, g') and P, F, t, n the smooth maps associated with (N, f), that are defined is Section 2. We recall that the mixed covariant derivatives $\overline{\nabla}F$, $\overline{\nabla}t$ are defined by

$$\begin{aligned}(\overline{\nabla}_X F)Y &= \nabla^\perp_X FY - F(\nabla'_X Y),\\ (\overline{\nabla}_X t)V &= \nabla'_X tV - t(\nabla^\perp_X V),\end{aligned}$$

for any $X, Y \in \Gamma(TN)$, $V \in \Gamma(T^\perp N)$.
We are going to relate the covariant derivatives $\nabla' P$, $\overline{\nabla} F$, $\overline{\nabla} t$, $\nabla^\perp n$ to the second fundamental form h and the Weingarten operators A_V. Hereinafter, if there is no danger of confusion, we will denote the metric g' by g, again.

Proposition 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold and (N, f) a submanifold of M such that ξ is normal to N. Then, for any $X, Y \in \Gamma(TN)$, $V \in \Gamma(T^\perp N)$, one has*

i) $A_\xi X = -(\alpha \circ f)X, \quad \nabla^\perp_X \xi = 0,$

ii) $(\nabla'_X P)Y = A_{FY}X + th(X, Y),$

iii) $(\overline{\nabla}_X F)Y = nh(X, Y) - h(X, PY) + (\alpha \circ f)g(PX, Y)\xi,$

iv) $(\overline{\nabla}_X t)V = A_{nV}X - P(A_V X) - (\alpha \circ f)\eta(V)PX,$

v) $(\nabla^\perp_X n)V = -h(X, tV) - F(A_V X) + (\alpha \circ f)\{g(FX, V)\xi - \eta(V)FX\}.$

Proof. By (4.1), for any $X \in \Gamma(TN)$ we have $\nabla_X \xi = (\alpha \circ f)X$. Then *i)* follows applying the Weingarten equation. Given $X, Y \in \Gamma(TN)$, using (4.1), the Gauss and Weingarten equations, we have

$$\begin{aligned}(\alpha \circ f)g(PX, Y)\xi = (\nabla_X \varphi)Y =& (\nabla'_X P)Y - A_{FY}X - th(X, Y) + (\overline{\nabla}_X F)Y \\ &+ h(X, PY) - nh(X, Y).\end{aligned}$$

Comparing the tangential and normal components, we get *ii)*, *iii)*. Analogously, considering $V \in \Gamma(T^\perp N)$, $X \in \Gamma(TN)$, one has

$$\begin{aligned}&(\alpha \circ f)\{g(FX, V)\xi - \eta(V)PX\} - (\alpha \circ f)\eta(V)FX = (\nabla_X \varphi)V \\ &\quad = (\overline{\nabla}_X t)V - A_{nV}X + P(A_V X) + (\nabla^\perp_X n)V + h(X, tV) + F(A_V X).\end{aligned}$$

Then, *iv)*, *v)* follow. □

The next result provides a characterization of slant submanifolds of any a.c.m. manifold which involves the behavior of the tensor fields $Q = P^2$ and $\nabla' Q$.

Proposition 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be an a.c.m. manifold and (N, f) a submanifold of M such that ξ is normal to N. The following conditions are equivalent*

i) *(N, f) is slant.*

ii) *$\nabla' Q = 0$ and at any point of N the endomorphism Q admits only one eigenvalue.*

Moreover, if (N, f) is slant with sla$(N) = \theta$, the unique eigenfunction λ of Q is constant and $\lambda = -\cos^2\theta$.

Proof. If (N, f) is slant, then $Q = (-\cos^2\theta)I_{TN}$ is parallel and the only eigenfunction of Q is the constant function $-\cos^2\theta$.

Conversely, assume that $\nabla' Q = 0$ and $Q = \lambda I_{TN}$. For any $X, Y \in \Gamma(TN)$ one has $0 = (\nabla'_X Q)Y = X(\lambda)Y$. It follows that $\lambda \colon N \to [-1, 0]$ is a constant function. Hence, there exists $\theta \in [0, \frac{\pi}{2}]$ such that $\lambda = -\cos^2\theta$ and (N, f) is slant. □

Corollary 4.1. *Let (N, f) be a slant submanifold of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ such that ξ is normal to N. For any $X, Y \in TN$, we have*

$$A_{FPX}Y + A_{FPY}X - A_{FX}PY - A_{FY}PX = 0.$$

Proof. By Propositions 4.1, 4.2, for any $Y, Z \in \Gamma(TN)$ we have

$$\begin{aligned} 0 = (\nabla'_Z Q)Y &= (\nabla'_Z P)PY + P((\nabla'_Z P)Y) \\ &= A_{FPY}Z + th(Z, PY) + P(A_{FY}Z) + Pth(Z, Y). \end{aligned}$$

Thus, for any $X, Y, Z \in TN$ we obtain

$$\begin{aligned} 0 &= g(A_{FPY}Z, X) - g(h(Z, PY), FX) - g(A_{FY}Z, PX) + g(h(Z, Y), FPX) \\ &= g(A_{FPY}X - A_{FX}PY - A_{FY}PX + A_{FPX}Y, Z). \end{aligned}$$

□

Theorem 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold and (N, f) a slant submanifold of M such that $sla(N) = \theta \neq \frac{\pi}{2}$ and ξ is normal to N. Then, the a.H. structure induced on N by f is almost Kähler.*

Proof. We consider the a.H. structure $(J = \frac{1}{\cos\theta}P, g)$ on N and prove that the fundamental 2-form Ω, $\Omega(X, Y) = g(X, JY)$, is closed. In fact, applying Proposition 4.1, for any $X, Y, Z \in \Gamma(TN)$ one has

$$\begin{aligned} d\Omega(X, Y, Z) &= -\frac{1}{3} \underset{(X,Y,Z)}{\sigma} g((\nabla'_X J)Y, Z) \\ &= -\frac{1}{3\cos\theta} \underset{(X,Y,Z)}{\sigma} g(A_{FY}X + th(X, Y), Z) \\ &= -\frac{1}{3\cos\theta} \underset{(X,Y,Z)}{\sigma} g(A_{FY}Z - A_{FZ}Y, X) = 0. \end{aligned} \tag{4.2}$$

□

Corollary 4.2. *In the same hypotheses of* Theorem 4.1, *for any $X, Y \in TN$ one has*

$$P(A_{FY}X - A_{FX}Y) + A_{FPY}X - A_{FX}PY = 0.$$

Proof. By Theorem 4.1, (N, J, g) is an almost Kähler manifold, therefore for any $Y, Z \in TN$ we have $(\nabla'_Z J)Y + (\nabla'_{JZ} J)JY = 0$ ([9]). Hence, also applying Proposition 4.1, for any $X, Y, Z \in TN$ we have

$$\begin{aligned} 0 &= g((\nabla'_Z P)Y + (\nabla'_{JZ} P)JY, X) \\ &= g(A_{FY}Z + th(Y, Z), X) + \frac{1}{\cos^2\theta} g(A_{FPY}PZ + th(PY, PZ), X) \\ &= g(A_{FY}X - A_{FX}Y, Z) - \frac{1}{\cos^2\theta} g(P(A_{FPY}X - A_{FX}PY), Z). \end{aligned}$$

It follows that

$$(\cos^2\theta)(A_{FY}X - A_{FX}Y) - P(A_{FPY}X - A_{FX}PY) = 0$$

and, applying P, we obtain the statement. □

Remark 4.1. We recall that any a.H. structure on a 2-dimensional manifold is a Kähler structure. Hence, considering a submanifold (N, f) as in Theorem 4.1 with $\dim N = 2$, we get that $(N, \frac{1}{\cos\theta}P, g)$ is a Kähler manifold and Proposition 4.1 entails that, for any $X, Y \in TN$ $A_{FX}Y = A_{FY}X$. It follows that Corollary 4.2 is trivial.

We observe that any submanifold (N, f) of a $C_5 \oplus C_{12}$-manifold M such that $\xi \in \Gamma(T^\perp N)$ satisfies the condition $A_\xi \circ P = P \circ A_\xi$. Now, we are going to study slant submanifolds for which all the Weingarten operators A_{FX} commute with P.

Theorem 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold and (N, f) a slant submanifold of M such that $sla(N) = \theta \neq \frac{\pi}{2}$ and ξ is normal to N. Assume that for any $X \in TN$ $A_{FX} \circ P = P \circ A_{FX}$. Then, all the Weingarten operators A_{FX}, $X \in TN$, vanish and $(N, J = \frac{1}{\cos\theta}P, g)$ is a Kähler manifold.*

Proof. We know that (N, J, g) is an almost Kähler manifold. Hence, applying Proposition 4.1, for any $X, Y \in TN$ we have

$$A_{FY}X + th(X, Y) + A_{FJY}JX + th(JX, JY) = 0.$$

By this equation, taking the skew-symmetric component, we get

$$A_{FY}X - A_{FX}Y + A_{FJY}JX - A_{FJX}JY = 0.$$

Now, we apply J and the hypothesis, so obtaining

$$\begin{aligned} A_{FY}JX - A_{FX}JY - A_{FJY}X + A_{FJX}Y &= J(A_{FY}X - A_{FX}Y \\ + A_{FJY}JX - A_{FJX}JY) &= 0. \end{aligned}$$

Then, by Corollary 4.1, one gets $A_{FX}JY = A_{FJX}Y$ and, for any Z tangent to N, we obtain

$$\begin{aligned} g(Jth(X, Y), Z) &= g(h(X, Y), FJZ) = g(A_{FZ}JX, Y) \\ &= -g(A_{FZ}X, JY) = g(th(X, JY), Z). \end{aligned}$$

It follows

$$th(X, JY) = Jth(X, Y) = th(Y, JX), \quad X, Y \in TN. \tag{4.3}$$

On the other hand, using the hypothesis, for any $X, Y, Z \in TN$ we obtain

$$g(th(JX, Y), Z) = -g(A_{FZ}JX, Y) = g(A_{FZ}X, JY) = -g(th(X, JY), Z).$$

This implies $th(JX, Y) = -th(X, JY)$ and, combining with (4.3), for any $X, Y \in TN$ we have $th(X, Y) = 0$, $A_{FX} = 0$. This proves the statement. □

5 Slant immersions and second fundamental form

Let (N, f) be a submanifold of an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ such that ξ is normal to N and denote by μ the vector subbundle of $T^\perp N$ whose fibre, at any $x \in N$, is the orthogonal complement to $< \xi_x > \oplus F(T_xN)$ in $T^\perp_x N$. So we can consider the orthogonal splitting

$$TM_{|N} = TN \oplus < \xi > \oplus F(TN) \oplus \mu. \tag{5.1}$$

Lemma 5.1. *Let (N, f) be a submanifold of an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ such that ξ is normal to N. For any $V \in \Gamma(T^{\perp}N)$, we have*

i) $V \in \Gamma(\mu) \Rightarrow \varphi V \in \Gamma(\mu)$.

ii) $V \in \Gamma(< \xi > \oplus \mu) \Leftrightarrow \varphi V = nV \Leftrightarrow tV = 0$.

Proof. If $V \in \Gamma(\mu)$, by (2.1), (5.1), for any $X \in \Gamma(TN)$ one has $g(\varphi V, X) = -g(V, FX) = 0$, $g(\varphi V, FX) = -g(V, nFX) = g(V, FPX) = 0$. Then *i)* follows.

If $V \in \Gamma(< \xi > \oplus \mu)$, then for any $X \in \Gamma(TN)$ we have $g(tV, X) = -g(V, FX) = 0$. It follows $tV = 0$, equivalently $\varphi V = nV$.
Conversely, if $\varphi V = nV$, for any X tangent to N one has $g(V - \eta(V)\xi, FX) = g(V, FX) = -g(nV, X) = 0$. It follows that $V - \eta(V)\xi \in \Gamma(\mu)$, that is, $V \in \Gamma(< \xi > \oplus \mu)$. □

We observe that, if (N, f) is a proper slant submanifold of M such that $\dim N = 2n$, then $\mathrm{rank} F(TN) = 2n$. Applying (5.1) it follows that $\dim M \geq 4n + 1$ and $\dim M = 4n + 1$ if and only if μ is trivial.

The next result puts in evidence the interplay between μ and the second fundamental form.

Theorem 5.1. *Let (N, f) be a proper slant submanifold of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ such that $sla(N) = \theta$ and $\dim N = 2n$. Assume that for any $V \in T^{\perp}N$ $A_V \circ P = P \circ A_V$. Then, the following properties hold*

i) If $\dim M = 4n + 1$, (N, f) is totally umbilical in M and $H = -(\alpha\xi)_{|N}$ is its mean curvature vector field.

ii) If $\dim M > 4n + 1$ and $\nabla^{\perp}_X(h(Y, Z)) \in \Gamma(< \xi > \oplus \mu)$ for any $X, Y, Z \in \Gamma(TN)$, then (N, f) is totally umbilical in M with mean curvature vector field $H = -(\alpha\xi)_{|N}$.

Moreover, in both cases i) and ii), if (N, f) is not totally geodesic, then H is parallel if and only if $\nabla_\xi \xi \in \Gamma(F(TN) \oplus \mu)$.

Proof. By Theorem 4.2, for any $X \in TN$ we have $A_{FX} = 0$, equivalently $th(X, Y) = 0$, for any $X, Y \in \Gamma(TN)$. Therefore, Lemma 5.1 entails that $h(X, Y) \in \Gamma(< \xi > \oplus \mu)$.

If $\dim M = 4n + 1$, since μ is trivial, by Proposition 4.1 we have $h(X, Y) = g(h(X, Y), \xi)\xi = -(\alpha \circ f)g(X, Y)\xi$ and *i)* holds.

Now, we suppose that $\dim M > 4n + 1$ and, for any $X, Y, Z \in \Gamma(TN)$, $\nabla^{\perp}_X(h(Y, Z)) \in \Gamma(< \xi > \oplus \mu)$. By Proposition 4.1, for any $X, Y, Z, W \in \Gamma(TN)$ one has

$$\begin{aligned} g((\overline{\nabla}_X F)Y, nh(Z, W)) =& g(h(X, Y), h(Z, W)) - (\alpha \circ f)^2 g(X, Y)g(Z, W) \\ & - g(h(X, PY), nh(Z, W)). \end{aligned} \tag{5.2}$$

On the other hand, being $h(Z, W) \in \Gamma(< \xi > \oplus \mu)$, we also have $nh(Z, W) = \varphi(h(Z, W)) \in \Gamma(\mu)$ as well as $n(\nabla^{\perp}_X(h(Z, W))) = \varphi(\nabla^{\perp}_X(h(Z, W))) \in \Gamma(\mu)$. By Proposition 4.1 and (2.2) we have

$$\begin{aligned} g((\overline{\nabla}_X F)Y, nh(Z, W)) &= g(\nabla^{\perp}_X FY, nh(Z, W)) = -g(FY, (\nabla^{\perp}_X n)h(Z, W)) \\ &= (\sin^2\theta)\{g(h(X, Y), h(Z, W)) \\ &\qquad - (\alpha \circ f)^2 g(X, Y)g(Z, W)\}. \end{aligned}$$

Comparing with (5.2) we obtain

$$g(h(X,PY),nh(Z,W)) = (\cos^2\theta)\{g(h(X,Y),h(Z,W)) \\ -(\alpha\circ f)^2 g(X,Y)g(Z,W)\}. \tag{5.3}$$

Since the right hand side in (5.3) is symmetric with respect to X, Y, we get

$$g(h(X,PY),nh(Z,W)) = g(h(PX,Y),nh(Z,W)).$$

On the other hand, we have

$$g(h(X,PY),nh(Z,W)) = g(A_{nh(Z,W)}PY,X) \\ = -g(A_{nh(Z,W)}Y,PX) = -g(h(PX,Y),nh(Z,W)).$$

It follows $g(h(X,PY),nh(Z,W)) = 0$ and (5.3) implies that

$$g(h(X,Y),h(Z,W)) = (\alpha\circ f)^2 g(X,Y)g(Z,W).$$

In particular, for any $X, Y \in \Gamma(TN)$ one has $||h(X,Y)||^2 = (\alpha\circ f)^2 g(X,Y)^2 = g(h(X,Y),\xi)^2$, so that $h(X,Y) = g(h(X,Y),\xi)\xi = -(\alpha\circ f)g(X,Y)\xi$. Hence (N,f) is totally umbilical and the mean curvature vector field is $H = -(\alpha\xi)_{|N}$.

Finally, since $\dim M \geq 5$, for any $X \in \Gamma(TN)$ we have

$$\nabla^\perp_X H = -X(\alpha\circ f)\xi = -(\alpha\circ f)g(\nabla_\xi\xi, X)\xi.$$

Taking into account (5.1), it follows that if (N,f) is not totally geodesic, namely if $\alpha\circ f \neq 0$, then H is parallel if and only if $\nabla_\xi\xi \in \Gamma(F(TN)\oplus\mu)$. □

Remark 5.1. Let (M,φ,ξ,η,g) be a $C_5\oplus C_{12}$-manifold. If (N,f) is an invariant submanifold of M with $\dim N = 2n$, we have $T^\perp N =<\xi>\oplus\mu$ and $\dim M \geq 2n+1$. It follows that either (N,f) is an hypersurface of M or $\dim M > 2n+1$. In the first case, it is obvious that (N,f) is totally umbilical in M and $H = -(\alpha\xi)_{|N}$. Otherwise, we also assume that all the Weingarten operators commute with $P = \varphi_{|TN}$. Using the same technique as in Theorem 5.1, one proves that (N,f) is totally umbilical, $H = -(\alpha\xi)_{|N}$ and, if (N,f) is not totally geodesic, then H is parallel if and only if $\nabla_\xi\xi \in \Gamma(\mu)$.

Now, we explain some consequences of Theorems 4.2, 5.1.

Corollary 5.1. *Let (N,f) be a proper slant submanifold of a $C_5\oplus C_{12}$-manifold (M,φ,ξ,η,g) such that $\dim N = 2n$, $\dim M = 4n+1$. The following properties are equivalent*

i) For any $X \in TN$, $A_{FX}\circ P = P\circ A_{FX}$.

ii) (N,f) is totally umbilical in M.

Moreover, if one of the previous conditions holds, the mean curvature vector field of (N,f) is $H = -(\alpha\xi)_{|N}$.

Proof. The statement *i)* ⇒ *ii)* follows by Theorem 5.1.

Conversely, we assume that $h(X,Y) = g(X,Y)H$. For any $X, Y, Z \in TN$ we have

$$g(A_{FX}PY,Z) = g(H,FX)g(PY,Z) = -g(H,FX)g(Y,PZ) = g(P(A_{FX}Y),Z).$$

Thus, the Weingarten operators commute with P.

The last part of the statement follows by Theorem 5.1. □

Corollary 5.2. *Let (N, f) be a proper slant submanifold of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ such that* $\dim N = 2n$ *and* $\dim M > 4n + 1$*. If (N, f) is totally umbilical in M with mean curvature vector field H, one has*

i) For any $V \in T^{\perp}N$, $A_V \circ P = P \circ A_V$ and $H \in \Gamma(< \xi > \oplus \mu)$.

ii) For any $X \in \Gamma(TN)$, $\nabla^{\perp}_X H \in \Gamma(< \xi > \oplus \mu)$ if and only if $H = -(\alpha\xi)_{|_N}$.

Proof. As in Corollary 5.1, for any $V \in T^{\perp}N$ one has $A_V \circ P = P \circ A_V$. By Theorem 4.2 it follows that for any $X \in TN$ $A_{FX} \equiv 0$, and then $tH \equiv 0$. Therefore $H \in \Gamma(< \xi > \oplus \mu)$. By Proposition 4.1 we have $g(H, \xi)X = A_\xi X = -(\alpha \circ f)X$, for any $X \in TN$. This implies $\eta(H) = -(\alpha \circ f)$. Applying Proposition 4.1 again, for any $X \in \Gamma(TN)$ one has

$$t(\nabla^{\perp}_X H) = -(\overline{\nabla}_X t)H = P(A_H X) + (\alpha \circ f)\eta(H)PX = (||H||^2 - (\alpha \circ f)^2)PX.$$

It follows that for any $X \in \Gamma(TN)$ $\nabla^{\perp}_X H \in \Gamma(< \xi > \oplus \mu)$ if and only if $||H||^2 = (\alpha \circ f)^2 = g(H, \xi)^2$ if and only if $H = -(\alpha\xi)_{|_N}$. □

The next result is a consequence of Theorem 5.1 and Corollaries 5.1, 5.2.

Proposition 5.1. *Let (N, f) be a proper slant submanifold of a C_{12}-manifold $(M, \varphi, \xi, \eta, g)$ such that* $\dim N = 2n$*. The following properties hold*

i) If $\dim M = 4n + 1$*, then (N, f) is totally geodesic in M if and only if for any $X \in TN$ $A_{FX} \circ P = P \circ A_{FX}$.*

ii) If $\dim M > 4n + 1$*, then (N, f) is totally geodesic in M if and only if for any $V \in T^{\perp}N$ $A_V \circ P = P \circ A_V$ and for any $X, Y, Z \in \Gamma(TN)$ $\nabla^{\perp}_X(h(Y, Z)) \in \Gamma(< \xi > \oplus \mu)$.*

Now, we consider a submanifold (N, f) of a $C_5 \oplus C_{12}$-manifold such that the Reeb vector field ξ is normal to N. By Proposition 4.1, one easily obtains that, if (N, f) is totally umbilical with mean curvature vector field $H = -(\alpha\xi)_{|_N}$, then $\nabla' P = \overline{\nabla} F = \overline{\nabla} t = \nabla^{\perp} n = 0$. In order to see if the converse statement holds, we characterize the condition $\overline{\nabla} F = 0$.

Proposition 5.2. *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold and (N, f) a submanifold of M such that ξ is normal to N. The following conditions are equivalent*

i) $\overline{\nabla} F = 0$.

ii) For any $V \in T^{\perp}N$, $A_{nV} = -A_V \circ P - (\alpha \circ f)\eta(V)P$.

iii) For any $V \in T^{\perp}N$, $A_{nV} = P \circ A_V + (\alpha \circ f)\eta(V)P$.

iv) $\overline{\nabla} t = 0$.

Moreover, if F is parallel and (N, f) is slant with $sla(N) = \theta \neq \frac{\pi}{2}$, for any $X, Y \in TN$ we have

$$h(X, Y) + \frac{1}{\cos^2\theta} h(PX, PY) = -2(\alpha \circ f)g(X, Y)\xi \tag{5.4}$$

and $H = -(\alpha\xi)_{|_N}$ is the mean curvature vector field of (N, f).

Proof. By Proposition 4.1, for any $X, Y \in \Gamma(TN)$, $V \in \Gamma(T^\perp N)$ we have

$$g((\overline{\nabla}_X F)Y, V) = g((\alpha \circ f)\eta(V)PX + P(A_V X) - A_{nV}X, Y) = -g((\overline{\nabla}_X t)V, Y).$$

On the other hand, it is easy to prove the relation

$$\begin{aligned} & g(A_{nV}X + A_V(PX) + (\alpha \circ f)\eta(V)PX, Y) \\ & = g(A_{nV}Y - P(A_V Y) - (\alpha \circ f)\eta(V)PY, X). \end{aligned}$$

Hence, the required equivalences hold. Finally, we assume that $\text{sla}(N) = \theta \neq \frac{\pi}{2}$ and $\overline{\nabla} F = 0$. Therefore, $P^2 = (-\cos^2\theta)I_{TN}$, $\dim N = 2n$ and, by Proposition 4.1, (2.2), for any $X, Y \in \Gamma(TN)$ we have

$$\begin{aligned} 0 = (\overline{\nabla}_Y F)PX - (\overline{\nabla}_{PX} F)Y = & (\cos^2\theta)h(X, Y) + h(PX, PY) \\ & + 2\cos^2\theta(\alpha \circ f)g(X, Y)\xi. \end{aligned}$$

This entails (5.4) and, considering a local orthonormal frame $\{e_i, e_{n+i} = \frac{Pe_i}{\cos\theta}\}$ on N, one has

$$\sum_{i=1}^{n}\{h(e_i, e_i) + h(e_{n+i}, e_{n+i})\} = -2n(\alpha \circ f)\xi.$$

It follows that $-(\alpha\xi)_{|N}$ is the mean curvature vector field of (N, f). □

Proposition 5.3. *Let (N, f) be a proper slant submanifold of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ such that ξ is normal to N and $\nabla^\perp n = 0$. The following properties hold*

i) For any $X, Y \in \Gamma(TN)$, $h(X, Y) \in \Gamma(<\xi> \oplus F(TN))$.

ii) The a.H. structure induced on N by f is a Kähler structure.

iii) For any $V \in \Gamma(\mu)$, $A_V = 0$.

iv) $\overline{\nabla} t = 0$ if and only if for any $X \in TN$ $A_{FX} \circ P = -P \circ A_{FX}$.

Proof. Let $\text{sla}(N) = \theta$. By Proposition 4.1, (2.1), (2.2) for any $X, Y \in \Gamma(TN)$ we have

$$0 = (\nabla^\perp_X n)FY = (\sin^2\theta)\{h(X, Y) + (\alpha \circ f)g(X, Y)\xi\} - F(A_{FY}X).$$

It follows

$$h(X, Y) + (\alpha \circ f)g(X, Y)\xi = \frac{1}{\sin^2\theta}F(A_{FY}X). \tag{5.5}$$

So, we obtain *i)* and, applying again Proposition 4.1, (2.1) we get

$$(\nabla'_X P)Y = th(X, Y) + A_{FY}X = \frac{1}{\sin^2\theta}tF(A_{FY}X) + A_{FY}X = 0.$$

Thus $J = \frac{1}{\cos\theta}P$ is parallel and *ii)* holds. By Proposition 4.1, for any $V \in \Gamma(\mu)$, $X \in \Gamma(TN)$ we have $0 = (\nabla^\perp_X n)V = -F(A_V X)$, hence $A_V X = 0$. Therefore *iii)* is proved. Note that, by Lemma 5.1, for any $V \in \Gamma(\mu)$ we also have $A_{nV} = 0$ and then $(\overline{\nabla}_X t)V = 0$. Moreover $(\overline{\nabla}_X t)\xi = -P(A_\xi X) - (\alpha \circ f)PX = 0$. This implies that $\overline{\nabla} t = 0$ if and only if $(\overline{\nabla}_X t)FY = 0$, for any $X, Y \in \Gamma(TN)$. We also remark that, being $\nabla' P = 0$, for any $X, Y \in TN$ one has $A_{FX}Y = A_{FY}X$ and Corollary 4.2 entails $A_{FPY}X = A_{FX}PY$. Using Proposition 4.1 and (2.1) we obtain

$$(\overline{\nabla}_X t)FY = -A_{FPY}X - P(A_{FY}X) = -A_{FX}PY - P(A_{FX}Y).$$

Thus, *iv)* is proved. □

Proposition 5.4. *Let (N, f) be a proper slant submanifold of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ such that ξ is normal to N and $\overline{\nabla} F = 0$. The following properties hold*

i) For any $V \in \Gamma(\mu)$, $A_V = 0$.

ii) $F(TN)$ is a parallel subbundle of $T^\perp N$.

iii) $\nabla^\perp n = 0$ if and only if the a.H. structure induced on N by f is a Kähler structure.

Proof. Since $\overline{\nabla} F = 0$, by Proposition 4.1 one has

$$g(h(X, Y), nV) + g(h(X, PY), V) = 0, \quad V \in \Gamma(\mu), X, Y \in \Gamma(TN). \tag{5.6}$$

So, also applying (2.1) and Lemma 5.1, for any $V \in \Gamma(\mu)$, $X, Y \in \Gamma(TN)$, we have

$$\begin{aligned} g(A_V X, Y) &= -g(h(X, Y), n^2 V) = g(h(X, PY), nV) \\ &= -g(h(X, P^2 Y), V) = (\cos^2 \theta) g(A_V X, Y), \end{aligned}$$

where $\mathrm{sla}(N) = \theta \neq 0$. This implies *i)*. Now, considering $X, Y \in \Gamma(TN)$, we have $g(\nabla^\perp_X FY, \xi) = 0$ and for any $V \in \Gamma(\mu)$ $g(\nabla^\perp_X FY, V) = g((\overline{\nabla}_X F)Y, V) = 0$. Therefore $\nabla^\perp_X FY \in \Gamma(F(TN))$ and *ii)* is proved. By direct calculus, applying Proposition 4.1 and *i)*, for any $X \in \Gamma(TN), V \in \Gamma(\mu)$ we have $(\nabla^\perp_X n)V = 0$. Since $\nabla^\perp_X \xi = 0$, we also obtain $(\nabla^\perp_X n)\xi = 0$. It follows that n is parallel if and only if for any $X, Y \in \Gamma(TN)$ $(\nabla^\perp_X n)FY = 0$. Now, we prove the following formula

$$(\sin^2 \theta)\{h(X, Y) + (\alpha \circ f) g(X, Y)\xi\} = -Fth(X, Y), \quad X, Y \in TN. \tag{5.7}$$

In fact, by (2.1), Proposition 4.1 and the hypothesis, given $X, Y \in TN$ one has

$$\begin{aligned} &Fth(X, Y) + h(X, Y) + (\alpha \circ f) g(X, Y)\xi = -n^2 h(X, Y) \\ &= -nh(X, PY) = -h(X, P^2 Y) + (\alpha \circ f) g(PX, PY)\xi \\ &= (\cos^2 \theta)\{h(X, Y) + (\alpha \circ f) g(X, Y)\xi\}. \end{aligned}$$

It follows $Fth(X, Y) + (1 - \cos^2 \theta)\{h(X, Y) + (\alpha \circ f) g(X, Y)\xi\} = 0$ and then (5.7). Finally, by Proposition 4.1, (2.1) and (5.7) for any $X, Y \in \Gamma(TN)$ we obtain

$$\begin{aligned} (\nabla^\perp_X n)FY &= (\sin^2 \theta)\{h(X, Y) + (\alpha \circ f) g(X, Y)\xi\} - F(A_{FY}X) \\ &= -F((\nabla'_X P)Y). \end{aligned}$$

It follows that $\nabla^\perp n = 0$ if and only if $\nabla' P = 0$. □

The next result is a direct consequence of Propositions 5.2, 5.3, 5.4.

Corollary 5.3. *Let (N, f) be a proper slant submanifold of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ such that ξ is normal to N, $\overline{\nabla} F = 0$ and $\nabla^{\perp} n = 0$. Then, $H = -(\alpha\xi)_{|N}$ is the mean curvature vector field of N, $(J = \frac{1}{\cos\theta} P, g)$ is a Kähler structure on N and $\overline{\nabla} t = 0$.*

Finally, we relate the condition $\overline{\nabla} F = 0$ to the property of austere submanifold.
Given a submanifold (N, f) of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(T^{\perp} N)$, we know that at any point $x \in N$ the only eigenvalue of A_ξ is $-\alpha(f(x))$. It follows that, if $\alpha \circ f \neq 0$, the set of eigenvalues of A_ξ is not invariant under multiplication by -1, so N cannot be austere. As in Section 2, for any $x \in N$, $X \in T_x N$, $X \neq 0$, we denote by $\theta(X) \in [0, \frac{\pi}{2}]$ the angle between φX and $T_x N$.

Proposition 5.5. *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold, (N, f) a submanifold of M such that ξ is normal to N, $\alpha \circ f = 0$ and for any $X \in TN$ the angle $\theta(X) \in [0, \frac{\pi}{2})$. If $\overline{\nabla} F = 0$, then (N, f) is austere.*

Proof. Since $\alpha \circ f = 0$ and F is parallel, applying Proposition 5.2 we have $A_V \circ P + P \circ A_V = 0$, for any $V \in T^{\perp} N$. Given $x \in N$, we consider $V \in T_x^{\perp} N$ such that $A_V \neq 0$ and an eigenvalue $\beta \neq 0$ of A_V. Thus, there exists $X \in T_x N$, $X \neq 0$, such that $A_V X = \beta X$. Since $\theta(X) \neq \frac{\pi}{2}$, we have $PX \neq 0$ and $A_V(PX) = -\beta PX$. It follows that $-\beta$ is an eigenvalue of A_V. □

Corollary 5.4. *Let (N, f) be a slant submanifold of a C_{12}-manifold such that $sla(N) = \theta \neq \frac{\pi}{2}$ and $\dim N = 2n$. If $\overline{\nabla} F = 0$, then (N, f) is austere.*

Remark 5.2. Given an open interval $I \subset \mathbb{R}$, $0 \in I$, for any $\lambda \in C^{\infty}(\mathbb{R}^4)$, $\lambda > 0$, the submanifold $(\mathbb{R}^2, f_\lambda)$ of the C_{12}-manifold ${}_{\lambda}I \times \mathbb{R}^4$ considered in Example 3.1 fulfills the hypothesis of Corollary 5.4. In fact, since $(\mathbb{R}^2, f_\lambda)$ is invariant, one has $F_\lambda \equiv 0$.

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References

[1] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics **203**, Birkhäuser Boston, Boston, 2010.

[2] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, M. Fernández, Slant submanifolds in Sasakian manifolds, *Glasg. Math. J.* **42** (2000), 125-138.

[3] A. Carriazo, L. M. Fernández, M. B. Hans-Uber, Some slant submanifolds of S-manifolds, *Acta Math. Hungar.* **107** (2005), 267-285.

[4] B. Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, Louvain, 1990.

[5] B. Y. Chen, Slant immersions, *Bull. Austral. Math. Soc.* **41** (1990), 135-147.

[6] D. Chinea, C. Gonzalez, A classification of almost contact metric manifolds, *Ann. Mat. Pura Appl.* (4) **156** (1990), 15-36.

[7] S. de Candia, M. Falcitelli, Slant immersions in C_5-manifolds, to appear on *Bull. Math. Soc. Sci. Math. Roumanie.*

[8] M. Falcitelli, A class of almost contact metric manifolds and double twisted products, *Mathematical Sciences and Applications E-Notes (MSAEN)* **1** (2013), 36-57.

[9] A. Gray, L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl.* (4) **123** (1980), 35-58.

[10] R. S. Gupta, S. M. Khursheed Haider, A. Sharfuddin, Slant submanifolds in cosymplectic manifolds, *Colloq. Math.* **105** (2006), 207-219.

[11] A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Sc. Math. Roumanie* **39** (87) (1996), 183-198.

[12] F. Malek, M. B. K. Balgeshir, Slant submanifolds of almost contact metric 3-structure manifolds, *Mediterr. J. Math.* **10** (2013), 1023-1033.

[13] R. Ponge, H. Reckziegel, Twisted products in pseudo-Riemannian geometry, *Geom. Dedicata* **48** (1993), 15-25.

[14] B. Shain, Slant submanifolds of an almost product Riemannian manifold, *J. Korean Math. Soc.* **43** (2006), 717-732.

[15] B. Shain, Slant submanifolds of quaternion Kaehler manifolds, *Commun. Korean Math. Soc.* **22** (2007), 123-135.

[16] K. Yano, M. Kon, *Structures on manifolds*, Series in Pure Mathematics **3**, World Scientific Publishing Co., Singapore, 1984.

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