On the fundamental group of compact homogeneous manifolds carrying an invariant fat distribution^{*}

A. Lotta

Abstract

We show that a homogeneous space of a compact Lie group carrying an invariant fat distribution must have finite fundamental group.

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1 Introduction

The aim of this note is to prove the following:

Theorem 1.1 *Let $M = G/H$ be a homogeneous space of a compact Lie group G . If M admits a G -invariant fat distribution, then $\pi_1(M)$ is finite.*

In order to explain the “fat” assumption, we recall the relevant notions involved. We shall denote by ΓE the module of smooth sections of a vector bundle E . Given a vector subbundle $D \subset TM$ of the tangent bundle of a smooth manifold M (or equivalently a smooth distribution), the corresponding *Levi-Tanaka form* $\mathfrak{L}_D$ is the skew-symmetric $C^\infty(M)$ -bilinear map

$$\mathfrak{L}_D : \Gamma D \times \Gamma D \rightarrow \Gamma(TM/D),$$

defined by

$$\mathfrak{L}_D(X, Y) := \pi[X, Y]$$

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where $\pi : TM \rightarrow TM/D$ is the canonical projection. The classical theorem of Frobenius states that D is an integrable distribution iff $\mathcal{L}_D = 0$. We recall that D is called *Levi nondegenerate* provided

$$(\mathcal{L}_D)_x : D_x \times D_x \rightarrow T_x M/D_x \text{ is nondegenerate for each } x \in M,$$

that is

$$\{v \in D_x : (\mathcal{L}_D)_x(v, w) = 0 \text{ for every } w \in D_x\} = \{0\}.$$

Now, by definition, a *fat* distribution is one for which the following stronger condition holds:

$$(\mathcal{L}_D)_x(v, -) \text{ is onto for each } x \in M \text{ and every nonzero } v \in D_x.$$

We refer the reader to [5] for a recent analysis of the differential geometry of fat distributions, along the line of research started in [9]. We also remark that this notion has been intensively used in the context of sub-Riemannian geometry, where the term often used is “strongly bracket generating”; indeed, it is equivalent to requiring that, for any smooth nonvanishing section X of D , ΓTM is spanned by X and $[X, D]$. See [8] and also [2].

If D has codimension one, fatness reduces to Levi nondegeneracy. As regards this case, we mention the following application of our result in the context of *CR* geometry:

Corollary 1.2 *Let (M, HM, J) be a connected, compact homogeneous CR manifold of hypersurface type with nondegenerate Levi form. Then $\pi_1(M)$ is finite.*

This has been already proved in [3] and, in the strongly pseudo convex case, in [4]. We also refer the reader to these papers for the basic notions of *CR* geometry. Our proof appears to be more elementary. In order to apply Theorem 1.1 to the holomorphic distribution $D = HM$, it suffices to use the fact that, under the assumptions of the corollary, M always admits a *compact* transitive group G of *CR* automorphisms: a direct proof of this statement has been provided in [7].

2 Proof of Theorem 1.1

Fix a bi-invariant metric g on G ; denote by $\langle \cdot, \cdot \rangle$ the corresponding $Ad(G)$ -invariant inner product on the Lie algebra $\mathfrak{g}$. We consider the induced

G -invariant normal metric on G/H , denoted by the same symbol, turning $(G/H, g)$ into a naturally reductive homogeneous Riemannian manifold with nonnegative sectional curvature (cf. e.g. [6, Corollary X.3.6]). We claim that the existence of a G -invariant fat distribution forces the Ricci tensor of g to be positive definite, so that (by homogeneity) Myers' Theorem applies yielding that $\pi_1(M)$ must be finite. To prove our claim, let $o = \pi(e)$, $\pi : G \rightarrow G/H$ being the canonical projection; the reductive decomposition of the Lie algebra $\mathfrak{g}$ of G is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad (1)$$

where $\mathfrak{h} = \text{Lie}(H)$ and $\mathfrak{m} = \mathfrak{h}^\perp$ is the orthogonal complement of $\mathfrak{h}$ with respect to $\langle \cdot, \cdot \rangle$; the Ricci tensor of g at o is given, modulo the standard identification $T_o M \cong \mathfrak{m}$, by

$$\text{Ric}_o(X, X) = \frac{1}{4} \sum_{i=1}^n \|[X, E_i]_{\mathfrak{m}}\|^2 + \sum_{i=1}^n \|[X, E_i]_{\mathfrak{h}}\|^2 \quad (2)$$

where $n = \dim(M)$ and $\{E_i\}$ is an orthonormal basis of $\mathfrak{m}$. This formula is easily checked via a straightforward computation using Proposition 3.4 in [6, Ch. X, p. 202].

Now, consider the canonical G -invariant connection ∇ on the homogeneous space G/H , relative to the reductive decomposition (1) (cf. [6, Ch. X, p. 192]). We recall that in this setting ∇ is a metric connection with totally skew-symmetric torsion, i.e. its torsion tensor T , considered as a $(0, 3)$ tensor field, is a 3-form (see also [10] or [1]). Moreover, T is determined at o by

$$T_o(X, Y) = -[X, Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m}$$

We shall denote by $\mathfrak{z}$ the null space of the skew-symmetric bilinear map $T_o : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$. Now, we consider a G -invariant distribution D on M . According to Proposition X.2.7 in [6], because of the G -invariance, D must be parallel with respect to ∇ . Hence, after the canonical identification

$$TM/D \cong D^\perp$$

allowed by the metric g , the Levi form $\mathfrak{L} : \Gamma D \times \Gamma D \rightarrow \Gamma D^\perp$ of D can be expressed by means of the torsion T as follows:

$$\mathfrak{L}(X, Y) = -PT(X, Y) \quad X, Y \in \Gamma D \quad (3)$$

where $P : TM \rightarrow D^\perp$ is the orthogonal projection with respect to g .

Assume henceforth that D is fat. Hence, denoting by $\mathfrak{s}$ the subspace D_o of $\mathfrak{m}$, the skew-symmetric bilinear map

$$\mathfrak{L}_o : \mathfrak{s} \times \mathfrak{s} \rightarrow \mathfrak{s}^\perp$$

satisfies that $\mathfrak{L}_o(X, -)$ is onto for each non null vector $X \in \mathfrak{s}$. We claim that this forces $\mathfrak{z} = \{0\}$.

According to (3), and using the fact that T is a 3-form, we get, for every $X, Y \in \mathfrak{s}$ and $\xi \in \mathfrak{s}^\perp$:

$$\langle \mathfrak{L}_o(X, Y), \xi \rangle = -\langle T_o(X, Y), \xi \rangle = -T_o(X, Y, \xi) = \langle T_o(X, \xi), Y \rangle. \quad (4)$$

First of all we observe that this implies

$$\mathfrak{z} \cap \mathfrak{s} = \{0\} = \mathfrak{z} \cap \mathfrak{s}^\perp.$$

Indeed, the first equality holds since $\mathfrak{L}_o$ is nondegenerate, and the second one because $\mathfrak{L}_o$ is onto. Now, fix $U \in \mathfrak{z}$ and write

$$U = X + \xi$$

with $X \in \mathfrak{s}$ and $\xi \in \mathfrak{s}^\perp$. Then we must have $T_o(X, \xi) = 0$, which by (4) yields

$$\langle \mathfrak{L}_o(X, Y), \xi \rangle = 0$$

for every $Y \in \mathfrak{s}$. This implies that either $X = 0$ or $\xi = 0$, because D is fat. In both cases, we conclude that $U = 0$. Hence, by (2), the Ricci tensor is positive definite as claimed.

References

- [1] I. AGRICOLA: The Srní lectures on non-integrable geometries with torsion, *Arch. Math. (Brno)* **42** (2006), suppl., 5–84.
- [2] D. M. ALMEIDA: On fat sub-Riemannian symmetric spaces in codimension three, *Differential Geom. Appl.* **24** (2006), no. 2, 178–190.
- [3] H. AZAD; A. HUCKLEBERRY; W. RICHTHOFER: Homogeneous CR-manifolds, *J. Reine Angew. Math.* **358** (1985), 125–154.
- [4] D. BURNS; S. SHNIDER: Spherical hypersurfaces in complex manifolds, *Invent. Math.* **33** (1976), no. 3, 223–246.

- [5] A. KAPLAN; M. SUBILS: On the equivalence problems for bracket-generating distributions. Hodge theory, complex geometry, and representation theory, 157–171, *Contemp. Math.* **608**, Amer. Math. Soc., Providence, RI, 2014.
- [6] S. KOBAYASHI; K. NOMIZU: Foundations of differential geometry. Vol II. Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney 1969.
- [7] A. SPIRO: Groups acting transitively on compact CR manifolds of hypersurface type, *Proc. Amer. Math. Soc.* **128** (2000), no. 4, 1141–1145.
- [8] R. S. STRICHARTZ: Sub-Riemannian geometry, *J. Differential Geom.* **24** (1986), no. 2, 221–263.
- [9] N. TANAKA: On the equivalence problems associated with simple graded Lie algebras, *Hokkaido Math. J.* **8** (1979), no. 1, 23–84.
- [10] F. TRICERRI; L. VANHECKE: Naturally reductive homogeneous spaces and generalized Heisenberg groups, *Compositio Math.* **52** (1984), no. 3, 389–408.

Author's address:

Antonio Lotta

Dipartimento di Matematica

Università degli Studi di Bari Aldo Moro

Via E. Orabona 4, 70125 Bari

ITALY

e-mail: `antonio.lotta@uniba.it`