$L^1 - L^1$ estimates for a doubly dissipative semilinear wave equation

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Abstract. In this paper, we derive energy estimates and $L^1 - L^1$ estimates, for the solution to the Cauchy problem for the doubly dissipative wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = 0, \quad t \geq 0, \; x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x). \end{cases}$$

The solution is influenced both by the diffusion phenomenon created by the damping term u_t, and by the smoothing effect brought by the damping term $-\Delta u_t$. Thanks to these two effects, we are able to obtain linear estimates which may be effectively applied to find global solutions in any space dimension $n \geq 1$, to the problems with power nonlinearities $|u|^p$, $|u_t|^p$ and $|\nabla u|^p$, in the supercritical cases, by only assuming small data in the energy space, and with L^1 regularity. We also derive optimal energy estimates and $L^1 - L^1$ estimates, for the solution to the semilinear problems.

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1. Introduction

In this paper, we derive energy estimates and $L^1 - L^1$ estimates, for the solution to the Cauchy problem for the wave equation with frictional and

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viscoelastic damping,

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = 0, & t \geq 0,\ x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), \end{cases} \tag{1.1}$$

then we apply these estimates to the semilinear problems with power nonlinearities $|u|^p$, $|u_t|^p$ and $|\nabla u|^p$. We find global existence of small data solutions to these semilinear problems, in any space dimension $n \geq 1$, for supercritical powers, and we derive energy estimates and $L^1 - L^1$ estimates.

The profile of the solution to (1.1) has been recently studied in [9], in the L^2 setting, by assuming data in the energy space, and in weighted L^1 spaces. The solution to (1.1) has many interesting properties, which allow us to employ techniques and obtain results which are new, if compared with the corresponding result for the wave equation with only frictional damping u_t, or with only viscoelastic damping $-\Delta u_t$.

The fact that there are two damping terms gives great benefits to the solution to (1.1). On the one hand, the solution to (1.1) inherits the same decay properties of the solution to the problem for the wave equation with only frictional damping u_t (see, in particular, [11]), which are sharp, thanks to the diffusion phenomenon (see [7] and, later, [6, 10, 14]). On the other hand, it has the same regularity of the solution to the problem for the wave equation with only viscoelastic damping $-\Delta u_t$, in particular, a smoothing effect appears for the time derivatives of the solution (see, in particular, [15]). In other words, the low-frequencies profile of the solution is modified by the presence of the damping u_t, and the high-frequencies profile of the solution is modified by the presence of the damping $-\Delta u_t$.

Thanks to the first property, one may effectively use sharp estimates for (1.1) to prove the global existence of small data solutions to the problem with power nonlinearity $|u|^p$, in the supercritical case $p > 1 + 2/n$, as it happens for the wave equation with only frictional damping (see, in particular, [16]). Thanks to the second property, one may obtain well-posedness results in L^q spaces, $L^1 - L^1$ estimates for the solution, and a smoothing effect for the time derivatives of the solution. We mention that only partial results are known for the wave equation with only viscoelastic damping and power nonlinearity $|u|^p$ (see [4]).

As a consequence of these properties combined together, we may obtain global existence to the problem with different power nonlinearities, in the supercritical case, in any space dimension $n \geq 1$, by only assuming small data in the energy space, with additional L^1 regularity. The corresponding result for the wave equation with only frictional damping and power nonlinearity $|u|^p$, only works in space dimensions $n = 1, 2$ (see [8]) and it can be extended only up to space dimension $n = 5$, using $L^p - L^q$ estimates [12]. The extension to any space dimension $n \geq 1$ requires stronger assumptions on the data (see, in particular, [16]).

In this paper, we show how these bounds on the space dimension can be removed, thanks to the presence of the viscoelastic damping $-\Delta u_t$. Also, we obtain sharp estimates for the L^1 norm of the solution to the semilinear problem and its time derivative. Similar properties have been proved for the wave equation with structural damping, $(-\Delta)^{\frac{1}{2}}u_t$, where $(-\Delta)^{\frac{1}{2}}$ is the fractional Laplace operator (see [1, 2, 13]), but in that case the structure of the solution is much simpler than the structure of the solution to (1.1) (for instance, the smoothing effect is stronger). More general results about evolution equations with structural damping and power nonlinearities $|u|^p$ and $|u_t|^p$ have been recently obtained in [3].

If we consider the problem with power nonlinearity $|u|^p$,

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = |u|^p, \quad t \geq 0, \ x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), \end{cases} \tag{1.2}$$

it is very easy to show that the critical exponent is Fujita exponent $1 + 2/n$ (the same of the problem without the viscoelastic damping). In particular, using the test function method, one immediately see that no global solutions to (1.2) may exist, if $p \in (1, 1 + 2/n]$, even in a weak sense (see, for instance, [5, 17]), under a suitable sign assumption on the initial data u_0, u_1.

To prove the global existence of small data solutions for $p > 1 + 2/n$, it is sufficient to use only energy estimates in space dimension $n = 1, 2$ (following as in [8]), but the use of suitable, optimal, $L^r - L^q$ linear estimates, in particular $L^1 - L^1$ estimates, allow us to obtain the result in any space dimension $n \geq 1$.

Theorem 1.1. *Let $n \geq 1$ and $p > 1+2/n$. Also, assume that $p \leq 1+2/(n-2)$, if $n \geq 3$. Then there exists $\varepsilon > 0$ such that for any data*

$$\begin{aligned} (u_0, u_1) \in \mathcal{A} \doteq (L^1 \cap H^1) \times (L^1 \cap L^2), \qquad \textit{with} \\ \|(u_0, u_1)\|_{\mathcal{A}} \doteq \|u_0\|_{L^1} + \|u_0\|_{H^1} + \|u_1\|_{L^1} + \|u_1\|_{L^2} \leq \varepsilon, \end{aligned} \tag{1.3}$$

there exists a unique solution

$$u \in \mathcal{C}([0, \infty), L^1 \cap H^1) \cap \mathcal{C}^1([0, \infty), L^1 \cap L^2) \tag{1.4}$$

to (1.2). *Moreover, it satisfies the following estimates*

$$\|\nabla u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{4} - \frac{1}{2}} \|(u_0, u_1)\|_{\mathcal{A}}, \tag{1.5}$$

$$\|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{4}} \|(u_0, u_1)\|_{\mathcal{A}}, \tag{1.6}$$

$$\|u_t(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{4} - 1} \|(u_0, u_1)\|_{\mathcal{A}}, \tag{1.7}$$

$$\|u(t, \cdot)\|_{L^1} \lesssim \|(u_0, u_1)\|_{\mathcal{A}}, \tag{1.8}$$

$$\|u_t(t, \cdot)\|_{L^1} \lesssim (1 + t)^{-1} \|(u_0, u_1)\|_{\mathcal{A}}. \tag{1.9}$$

The restrictions from above on p in Theorem 1.1 may be relaxed by assuming additional regularity on the data.

Our linear estimates also apply to the problem with power nonlinearity $|u_t|^p$:

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = |u_t|^p, \quad t \geq 0, \ x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x). \end{cases} \tag{1.10}$$

In this case, homogeneity arguments lead to expect that the critical exponent is 1. Indeed, we may prove global existence of small data solutions to (1.10), for $p>1$, in any space dimension $n\geq 1$, by only assuming small initial data as before.

Theorem 1.2. *Let $n=1$ and $p\in(1,2]$, or $n\geq 2$ and $1<p<1+2/n$. Then there exists $\varepsilon>0$ such that for any data as in* (1.3)*, there exists a unique solution as in* (1.4) *to* (1.10)*. Moreover, it satisfies estimates* (1.5)-(1.6)-(1.7)-(1.8)-(1.9)*.*

The restriction from above on p in Theorem 1.2 may be relaxed to $p\leq 2$ if $n=2,3$ and to $p<1+4/n$ if $n\geq 4$, by dropping the requirement that $\nabla u(t,\cdot)\in L^2$ (see the proof of Theorem 1.2).

Finally, we consider the problem with power nonlinearity $|\nabla u|^p$:

$$\begin{cases} u_{tt}-\Delta u+u_t-\Delta u_t=|\nabla u|^p, & t\geq 0,\ x\in\mathbb{R}^n,\\ (u,u_t)(0,x)=(u_0,u_1)(x). \end{cases} \tag{1.11}$$

In this case, we may prove global existence of small data solutions to (1.11), for $p>1+1/(n+1)$, in any space dimension $n\geq 1$. Clearly, to obtain an estimate for $\|\nabla u(t,\cdot)\|_{L^1}$, we also assume $\nabla u_0\in L^1$, when dealing with (1.11). Moreover, we ask $\Delta u_0\in L^2$, to get the same upper bound for p, that we have in Theorem 1.1.

Theorem 1.3. *Let $n\geq 1$ and $p>1+1/(n+1)$. Also, assume that $p\leq 1+2/(n-2)$, if $n\geq 3$. Then there exists $\varepsilon>0$ such that for any data*

$$\begin{aligned} (u_0,u_1)\in\mathcal{A}\doteq(W^{1,1}\cap H^2)\times(L^1\cap L^2), \qquad \textit{with}\\ \|(u_0,u_1)\|_{\mathcal{A}}\doteq\|u_0\|_{W^{1,1}}+\|u_0\|_{H^2}+\|u_1\|_{L^1}+\|u_1\|_{L^2}\leq\varepsilon, \end{aligned} \tag{1.12}$$

there exists a unique solution

$$u\in\mathcal{C}([0,\infty),W^{1,1}\cap H^2)\cap\mathcal{C}^1([0,\infty),L^1\cap L^2) \tag{1.13}$$

to (1.11)*. Moreover, it satisfies estimates* (1.6)-(1.7)-(1.8)-(1.9)*, and estimates*

$$\|\Delta u(t,\cdot)\|_{L^2}\lesssim(1+t)^{-\frac{n}{4}-1}\|(u_0,u_1)\|_{\mathcal{A}}, \tag{1.14}$$

$$\|\nabla u(t,\cdot)\|_{L^1}\lesssim(1+t)^{-\frac{1}{2}}\|(u_0,u_1)\|_{\mathcal{A}}. \tag{1.15}$$

Assuming also $\nabla u_0\in L^1$, it would become possible to construct a global solution to (1.2) and (1.10), which also verifies $\nabla u(t,\cdot)\in L^1$, and estimate (1.15). However, this additional regularity property for the solution is not essential to prove the global existence argument for problems (1.2) and (1.10), so we avoided to take extra assumptions on the data in the statements of Theorems 1.1 and 1.2. Of course, the assumption $\nabla u_0\in L^1$ becomes fundamental, when dealing with problem (1.11).

Notation. In this paper, we write $f\lesssim g$, when there exists a constant $C>0$ such that $f\leq Cg$. We write $f\approx g$ when $g\lesssim f\lesssim g$.

1.1. Main goal

The proofs of Theorems 1.1 and 1.3 are relatively standard, once we prove suitable linear estimates for (1.1), and we use a contraction argument and Duhamel's principle. However, the proof of Theorem 1.2 is more delicate.

First of all, the corresponding result is not true, in absence of the viscoelastic damping, since, in that case, regularity issues arise, when p is smaller than 2. To fix this problem, we use the smoothing effect appearing for the time derivative of the solution to (1.1), which is a consequence of the presence of the viscoelastic damping (see Theorem 2.2 in [15]). This smoothing effect lead to the employment of estimates, which are singular at $t = 0$, but whose singularity is integrable.

The use of singular estimates lead, in general, to a possible loss of decay, but we avoid it by using $L^m - L^2$ singular estimates, with $m = 2/p$. Indeed, the loss of decay, described by $t^{\frac{1}{m}-\frac{1}{2}}$, in Duhamel's integral, tends to 0 as $p \to 1$, i.e., to the critical exponent, and it is compensated by other contributions in the integral, when p is away from 1 (see Remark 3.1).

The description of this interesting effect, which is only attainable if both damping terms are present in the wave equation, was the first motivation for this paper.

Another motivation for this paper, is the objective to obtain optimal L^1 estimates, for linear and semilinear problems. Estimates in L^1 are, in general, more difficult to obtain than estimates in L^q, for $q > 1$. However, the special structure of the solution to (1.1) allow us to do that, in any space dimension $n \geq 1$. The use of $L^1 - L^1$ estimates also makes more elegant the argument employed to prove the global existence of small data solutions to the semilinear problems.

2. Linear estimates

After performing the Fourier transform with respect to x, $\hat{u} = \mathfrak{F}u$, the equation in (1.1) reads as

$$\hat{u}_{tt} + (1 + |\xi|^2)\hat{u}_t + |\xi|^2\hat{u} = 0. \tag{2.1}$$

Therefore, the characteristic roots of

$$\lambda^2 + (1 + |\xi|^2)\lambda + |\xi|^2 = 0, \quad \text{i.e.} \qquad (\lambda + 1)(\lambda + |\xi|^2) = 0,$$

are given by:

$$\lambda_- = -1, \qquad \lambda_+ = -|\xi|^2.$$

In particular, for any $|\xi| \neq 1$, the solution u may be decomposed in two components, $u = u_+ + u_-$, with

$$\begin{aligned} \hat{u}_- &= \frac{\lambda_+\hat{u}_0 - \hat{u}_1}{\lambda_+ - \lambda_-}\, e^{\lambda_- t} = \frac{-|\xi|^2\hat{u}_0 - \hat{u}_1}{1 - |\xi|^2}\, e^{-t}, \\ \hat{u}_+ &= \frac{-\lambda_-\hat{u}_0 + \hat{u}_1}{\lambda_+ - \lambda_-}\, e^{\lambda_+ t} = \frac{\hat{u}_0 + \hat{u}_1}{1 - |\xi|^2}\, e^{-t|\xi|^2}. \end{aligned}$$

In order to derive estimates for the solution to (1.1), it is crucial to study the behavior at low and high frequencies, namely, as $\xi \to 0$ and as $|\xi| \to \infty$.

With no loss of generality, we assume in the following that $\hat{u}_0$ and $\hat{u}_1$ vanish in a neighborhood of the unit sphere $S^{n-1} = \{|\xi| = 1\}$, and therefore so $\hat{u}$ itself does. Indeed, in any compact subset of $\mathbb{R}^n \setminus \{0\}$, one may immediately prove any of the estimates that we are going to prove in this Section, since an exponential decay appears for the solution localized at intermediate frequencies, and the regularity issues do not come into play in compact subsets of $\mathbb{R}^n \setminus \{0\}$ (see later, Remark 2.7).

Let $\Omega_0 = B_a(0) = \{|\xi| < a\}$, for some $a \in (0,1)$, and $\Omega_1 = \mathbb{R}^n \setminus \bar{B}_b(0)$, for some $b > 1$, be such that $\hat{u}_0, \hat{u}_1$ are supported in $\Omega_0 \cup \Omega_1$. With this assumption, we are legitimated to write $u = u^+ + u^-$.

First of all, we notice that there is not much to say about u^-, since:

$$u^-(t,x) = -e^{-t}\, w_0(x), \quad w_0 \doteq (1+\Delta)^{-1}(-\Delta u_0 + u_1) = \mathfrak{F}^{-1}\left(\frac{|\xi|^2 \hat{u}_0 + \hat{u}_1}{1-|\xi|^2}\right).$$

Clearly, w_0 is well-defined, thanks to the assumption that $\hat{u}_0$ and $\hat{u}_1$ vanish in a neighborhood of S^{n-1}. On the other hand, u^+ is the solution to the Cauchy problem for the heat equation

$$\begin{cases} v_t - \Delta v = 0, \\ v(0,x) = v_0(x), \end{cases} \tag{2.2}$$

with initial data

$$v_0 = (1+\Delta)^{-1}(u_0 + u_1) = \mathfrak{F}^{-1}\left(\frac{\hat{u}_0 + \hat{u}_1}{1-|\xi|^2}\right).$$

Briefly, we will write $u^+ = e^{t\Delta} v_0$. Clearly, v_0 is well-defined, thanks to the assumption that $\hat{u}_0$ and $\hat{u}_1$ vanish in a neighborhood of S^{n-1}.

This decomposition makes shorter the proof of the desired estimates for the solution to (1.1). First of all, we consider energy estimates.

Proposition 2.1. *The solution to* (1.1) *satisfies the following estimates:*

$$\|\Delta u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}\left(\|u_0\|_{H^2} + \|u_1\|_{L^2}\right), \tag{2.3}$$

$$\|\Delta u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}-1}\left(\|u_0\|_{L^1} + \|u_0\|_{H^2} + \|u_1\|_{L^1} + \|u_1\|_{L^2}\right), \tag{2.4}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}\left(\|u_0\|_{L^2} + \|u_1\|_{L^2}\right), \tag{2.5}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}-1}\left(\|u_0\|_{L^1} + \|u_0\|_{L^2} + \|u_1\|_{L^1} + \|u_1\|_{L^2}\right). \tag{2.6}$$

Also, for any $m \in [1,2]$, *such that*

$$n\left(\frac{1}{m} - \frac{1}{2}\right) < 1, \tag{2.7}$$

it satisfies the following estimates:

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|u_0\|_{L^m} + \|u_0\|_{H^1} + \|u_1\|_{L^m}\right), \tag{2.8}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}-\frac{3}{2}}\left(\|u_0\|_{L^1} + \|u_0\|_{H^1} + \|u_1\|_{L^1} + \|u_1\|_{L^m}\right), \tag{2.9}$$

whereas for any $m \in [1,2]$ *such that*

$$n\left(\frac{1}{m}-\frac{1}{2}\right) < 2, \tag{2.10}$$

it satisfies the following estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\big(\|u_0\|_{L^m}+\|u_0\|_{L^2}+\|u_1\|_{L^m}\big), \tag{2.11}$$

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}-1}\left(\|u_0\|_{L^1}+\|u_0\|_{L^2}+\|u_1\|_{L^1}+\|u_1\|_{L^m}\right). \tag{2.12}$$

Proof. First, we consider u^-. By Plancherel's theorem, one immediately obtains:

$$\|w_0\|_{H^\kappa} \lesssim \|u_0\|_{H^\kappa}+\|u_1\|_{L^2}, \qquad \kappa=0,1,2, \tag{2.13}$$

so that, $u^- = -e^{-t}w_0$ verifies estimates (2.3)-(2.4)-(2.5)-(2.6). Also, by Hölder's inequality,

$$\|(1-|\xi|^2)^{-1}|\xi|^j\hat{u}_1\|_{L^2} \lesssim \|\hat{u}_1\|_{L^{m'}} \lesssim \|u_1\|_{L^m},$$

for any $m \in [1,2]$, satisfying (2.7) if $j=1$, or (2.10) if $j=0$. Here $m' = m/(m-1)$ is the Hölder conjugate of m. Therefore, u^- also verifies (2.8)-(2.9)-(2.11)-(2.12).

Now we consider u^+. By using the Fourier transform mapping properties, and Hölder's inequality,

$$\begin{aligned}\|\partial_t^k\partial_x^\alpha u^+(t,\cdot)\|_{L^2} &\lesssim \||\xi|^{|\alpha|}\partial_t^k\hat{u}^+(t,\cdot)\|_{L^2}\\ &\lesssim \||\xi|^{|\alpha|+2k}e^{-t|\xi|^2}\|_{L^r(\Omega_0)}\left(\|\hat{u}_0\|_{L^{m'}(\Omega_0)}+\|\hat{u}_1\|_{L^{m'}(\Omega_0)}\right)\\ &\qquad + e^{-t}\left(\|\hat{u}_0\|_{L^2(\Omega_1)}+\|\hat{u}_1\|_{L^2(\Omega_1)}\right)\\ &\lesssim (1+t)^{-\frac{n}{2r}-\frac{|\alpha|}{2}-k}\left(\|u_0\|_{L^m}+\|u_0\|_{L^2}+\|u_1\|_{L^m}+\|u_1\|_{L^2}\right),\end{aligned}$$

for any $|\alpha|+2k=0,1,2$, and for any $m\in[1,2]$, where we set $r\in[1,2]$ such that

$$\frac{1}{r}=\frac{1}{2}-\frac{1}{m'}=\frac{1}{m}-\frac{1}{2}. \tag{2.14}$$

We remark that we used that $1/(1-|\xi|^2)$ is bounded in Ω_0 and $(1+|\xi|^2)/(1-|\xi|^2)$ is bounded in Ω_1. The exponential decay has been produced by using $e^{-t|\xi|^2}\leq e^{-t}$, for $\xi\in\Omega_1$, whereas, in the last line, the polynomial decay has been produced by using

$$\||\xi|^{|\alpha|+2k}e^{-t|\xi|^2}\|_{L^r(\Omega_0)} \lesssim \begin{cases}|\Omega_0|^{\frac{1}{r}} & \text{if } t\in[0,1],\\ t^{-\frac{n}{2r}-\frac{|\alpha|}{2}-k} & \text{if } t\geq 1,\end{cases}$$

i.e. using the well-known self-similarity of the fundamental solution to the heat equation, only for $t\geq 1$. That, is, for $t\geq1$ the change of variable $\eta=\sqrt{t}\,\xi$, gives:

$$\begin{aligned}\||\xi|^{|\alpha|+2k}e^{-t|\xi|^2}\|_{L^r(\Omega_0)} &\leq \||\xi|^{|\alpha|+2k}e^{-t|\xi|^2}\|_{L^r(\mathbb{R}^n)}\\ &= t^{-\frac{n}{2r}-\frac{|\alpha|}{2}-k}\,\||\eta|^{|\alpha|+2k}e^{-|\eta|^2}\|_{L^r(\mathbb{R}^n)}.\end{aligned}$$

The proofs of (2.3) and (2.5) follow by setting $m = 2$ (so that $r = \infty$), whereas the proofs of (2.4) and (2.6) follow by setting $m = 1$ (so that $r = 2$).

Now, let $m \in [1, 2]$ satisfy (2.7) or, respectively, (2.10), and r as in (2.14). To prove (2.8) or, respectively, (2.11), we estimate

$$\begin{aligned}\|\nabla^j u^+(t,\cdot)\|_{L^2} &\lesssim \||\xi|^j \hat{u}^+(t,\cdot)\|_{L^2}\\ &\lesssim \||\xi|^j e^{-t|\xi|^2}\|_{L^r(\Omega_0)} \big(\|\hat{u}_0\|_{L^{m'}(\Omega_0)} + \|\hat{u}_1\|_{L^{m'}(\Omega_0)}\big)\\ &\qquad + e^{-t} \big(\|\hat{u}_0\|_{L^{m'}(\Omega_1)} + \|\hat{u}_1\|_{L^{m'}(\Omega_1)}\big)\\ &\lesssim (1+t)^{-\frac{n}{2r}-\frac{j}{2}} \big(\|u_0\|_{L^m} + \|u_1\|_{L^m}\big),\end{aligned}$$

whereas, to prove (2.9) and (2.12), we estimate

$$\begin{aligned}\|\nabla^j u^+(t,\cdot)\|_{L^2} &\lesssim \||\xi|^j \hat{u}^+(t,\cdot)\|_{L^2}\\ &\lesssim \||\xi|^j e^{-t|\xi|^2}\|_{L^2(\Omega_0)} \big(\|\hat{u}_0\|_{L^\infty(\Omega_0)} + \|\hat{u}_1\|_{L^\infty(\Omega_0)}\big)\\ &\qquad + e^{-t} \big(\|\hat{u}_0\|_{L^{m'}(\Omega_1)} + \|\hat{u}_1\|_{L^{m'}(\Omega_1)}\big)\\ &\lesssim (1+t)^{-\frac{n}{4}-\frac{j}{2}} \big(\|u_0\|_{L^1} + \|u_0\|_{L^m} + \|u_1\|_{L^1} + \|u_1\|_{L^m}\big).\end{aligned}$$

We remark that we used (2.7) and, respectively, (2.10), to get $|\xi|^{-(2-j)} \in L^r(\Omega_1)$, with $j = 1$ and, respectively, $j = 0$.

This concludes the proof. □

Thanks to the very special structure of the solution to (1.1), we may also easily obtain $L^1 - L^1$ estimates.

Proposition 2.2. *The solution to* (1.1) *satisfies the following* $L^1 - L^1$ *estimates:*

$$\|u(t,\cdot)\|_{L^1} \lesssim \big(\|u_0\|_{L^1} + \|u_1\|_{L^1}\big), \tag{2.15}$$

$$\|\nabla u(t,\cdot)\|_{L^1} \lesssim (1+t)^{-\frac{1}{2}} \big(\|u_0\|_{W^{1,1}} + \|u_1\|_{L^1}\big), \tag{2.16}$$

$$\|u_t(t,\cdot)\|_{L^1} \lesssim (1+t)^{-1} \big(\|u_0\|_{L^1} + \|u_1\|_{L^1}\big). \tag{2.17}$$

Proof. First, we consider u^-. For any $|\xi| < 1$, we may write

$$\hat{w}_0 = \frac{|\xi|^2}{1-|\xi|^2} (\hat{u}_0 + \hat{u}_1) + \hat{u}_1,$$

whereas, for any $|\xi| > 1$, we may write:

$$\hat{w}_0 = -\hat{u}_0 - \frac{1}{|\xi|^2 - 1} (\hat{u}_0 + \hat{u}_1).$$

Therefore, by applying Lemmas A.1 and A.2, thanks to Young inequality, we derive:

$$\|w_0\|_{L^1} \lesssim \|u_0\|_{L^1} + \|u_1\|_{L^1}.$$

Let us consider ∇w_0. For any $|\xi| < 1$, we may write

$$i\xi\hat{w}_0 = \frac{|\xi|^2}{1-|\xi|^2} (i\xi\hat{u}_0) + \frac{i\xi}{1-|\xi|^2} \hat{u}_1,$$

whereas, for any $|\xi| > 1$, we may write:

$$i\xi\hat{w}_0 = -(i\xi\hat{u}_0) - \frac{1}{|\xi|^2 - 1}(i\xi\hat{u}_0) - \frac{i\xi}{|\xi|^2 - 1}\hat{u}_1.$$

By using again Lemmas A.1 and A.2, and Young inequality, we derive:

$$\|\nabla w_0\|_{L^1} \lesssim \|\nabla u_0\|_{L^1} + \|u_1\|_{L^1}.$$

Therefore, $u^- = -e^{-t}w_0(x)$ verifies the desired estimates. Now we consider $u^+ = e^{t\Delta}v_0$. Proceeding in a similar way, due to:

$$\begin{aligned}
\hat{v}_0 &= \hat{u}_0 + \hat{u}_1 + \frac{|\xi|^2}{1 - |\xi|^2}(\hat{u}_0 + \hat{u}_1),\\
i\xi\hat{v}_0 &= \frac{i\xi}{1 - |\xi|^2}(\hat{u}_0 + \hat{u}_1),
\end{aligned}$$

if $|\xi| < 1$, and

$$\begin{aligned}
\hat{v}_0 &= -\frac{1}{|\xi|^2 - 1}(\hat{u}_0 + \hat{u}_1),\\
i\xi\hat{v}_0 &= -\frac{i\xi}{|\xi|^2 - 1}(\hat{u}_0 + \hat{u}_1),
\end{aligned}$$

if $|\xi| > 1$, by using Young inequality, applying Lemmas A.1 and A.2, we obtain

$$\|v_0\|_{L^1} + \|\nabla v_0\|_{L^1} \lesssim \|u_0\|_{L^1} + \|u_1\|_{L^1}. \tag{2.18}$$

On the other hand, we write Δv_0 in the form

$$\begin{aligned}
-|\xi|^2\hat{v}_0 &= \frac{|\xi|^2}{|\xi|^2 - 1}(\hat{u}_0 + \hat{u}_1), \quad \text{if } |\xi| < 1,\\
-|\xi|^2\hat{v}_0 &= \hat{u}_0 + \hat{u}_1 + \frac{1}{|\xi|^2 - 1}(\hat{u}_0 + \hat{u}_1), \quad \text{if } |\xi| > 1,
\end{aligned}$$

so that we also obtain

$$\|\Delta v_0\|_{L^1} \lesssim \|u_0\|_{L^1} + \|u_1\|_{L^1}. \tag{2.19}$$

Estimates (2.15)-(2.16)-(2.17) follow for $u^+ = e^{t\Delta}v_0$ by the well-known $L^1 - L^1$ estimates for the solution to the heat equation, in particular:

$$\|\partial_t^k \nabla^j u^+(t,\cdot)\|_{L^1} \lesssim \begin{cases} \|\Delta^k \nabla^j v_0\|_{L^1}, & \text{for } t \in [0,1],\\ t^{-\frac{j}{2}-k}\|v_0\|_{L^1} & \text{for } t \geq 1, \end{cases}$$

for $j + k = 0, 1$. $\square$

Remark 2.3. Estimate (2.17) requires lesser regularity, with respect to u_0, than the corresponding estimate in [15], for the wave equation with only viscoelastic damping. This effect is related to the fact that the solution to our problem (1.1) decomposes in simpler terms, if compared with the corresponding ones in [15].

We also notice that estimate (2.19), which we used to prove (2.17), is no longer valid, in general, in space dimension $n \geq 2$, if we replace Δ by a different second order operator, like $\partial_{x_1}\partial_{x_2}$.

A peculiarity due to the presence of the viscoelastic damping, is the smoothing effect that appears for the time derivatives of the solution to (1.1). This smoothing effect may also be used at short time, if we allow singular estimates at $t = 0$ (see Theorem 2.2 in [15]). When studying semilinear problems, by using Duhamel's principle, it becomes possible to employ estimates which are singular at $t = 0$, provided that the singular power is integrable in 0. These singular estimates play a fundamental role in the proof of Theorem 1.2.

Proposition 2.4. *Let $u_0 = 0$. For any $m \in [1,2]$, such that* (2.10) *holds, the solution to* (1.1) *satisfies the following estimates, for any $t > 0$:*

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1}\, t^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u_1\|_{L^m}, \tag{2.20}$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-1}\, t^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|u_1\|_{L^1}+\|u_1\|_{L^m}\right). \tag{2.21}$$

Proof. The proof is analogous to the proof of (2.11)-(2.12), in Proposition 2.1, in particular, no change is needed for u_t^-. When we consider u_t^+, we use the smoothing effect of $e^{t\Delta}$ to produce a regularity gain, by paying a singular power at $t = 0$.

To prove (2.20), we estimate

$$\begin{aligned}
\|u_t^+(t,\cdot)\|_{L^2} &\lesssim \||\xi|^2\hat{u}^+(t,\cdot)\|_{L^2}\\
&\lesssim \||\xi|^2 e^{-t|\xi|^2}\|_{L^r(\Omega_0)}\,\|\hat{u}_1\|_{L^{m'}(\Omega_0)}\\
&\qquad + e^{-t/2}\,\|e^{-t|\xi|^2/2}\|_{L^r(\Omega_1)}\,\|\hat{u}_1\|_{L^{m'}(\Omega_1)}\\
&\lesssim (1+t)^{-1-\frac{n}{2r}}\,\|u_1\|_{L^m} + e^{-t/2}t^{-\frac{n}{2r}}\,\|u_1\|_{L^m},
\end{aligned}$$

where r is as in (2.14). We remark that we used that $|\xi|^2/(1-|\xi|^2)$ is bounded in Ω_1, and that $e^{-t|\xi|^2} \leq e^{-t/2}\,e^{-t|\xi|^2/2}$ in Ω_1. We used the self-similarity of the fundamental solution to the heat equation, namely, the change of variable $\eta = \sqrt{t}\,\xi$, to obtain the singular power:

$$\|e^{-t|\xi|^2/2}\|_{L^r(\Omega_1)} \leq \|e^{-t|\xi|^2/2}\|_{L^r(\mathbb{R}^n)} = t^{-\frac{n}{2r}}\,\|e^{-|\eta|^2/2}\|_{L^r(\mathbb{R}^n)}.$$

This concludes the proof of (2.20). The proof of (2.21) for u^+ follows from the previous step for $t \in [0,1]$, whereas we may directly estimate

$$\|u_t^+(t,\cdot)\|_{L^2} \lesssim \||\xi|^2\hat{u}^+(t,\cdot)\|_{L^2} \lesssim \||\xi|^2 e^{-t|\xi|^2}\|_{L^2}\,\|\hat{u}_1\|_{L^\infty} \lesssim t^{-\frac{n}{4}-1}\,\|u_1\|_{L^1},$$

for $t \geq 1$. This concludes the proof of (2.21). ☐

Remark 2.5. The decay rates obtained in the linear estimates are sharp, since $u \sim v$, with $v = e^{t\Delta}(u_0+u_1)$, as $t \to \infty$, in the sense that, for sufficiently smooth u_0, u_1, with suitable sign assumption on $u_0 + u_1$,

$$\|\partial_t^k\partial_x^\alpha u(t,\cdot)\|_{L^q} \sim \|\partial_t^k\partial_x^\alpha v(t,\cdot)\|_{L^q} \sim t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{|\alpha|}{2}-k},$$

the so-called *diffusion phenomenon.* The proof is based on the estimate of the low frequencies part of $u^+ - v$, since the other components of u and v exponentially decay as $t \to \infty$.

Remark 2.6. If we replace equation in (1.1) by

$$u_{tt} - \Delta u + a u_t - b\Delta u_t = 0, \qquad t \geq 0, \ x \in \mathbb{R}^n,$$

for some $a, b > 0$, the previous technique to derive linear estimates remain valid, since the behavior at $\xi \to 0$ and $|\xi| \to \infty$ remains unchanged. Indeed, the characteristic roots $\lambda_\pm(\xi)$, of

$$\lambda^2 + (a + b|\xi|^2)\lambda + |\xi|^2 = 0,$$

have the following behaviors:

$$\lambda_- \to \begin{cases} -a & \text{as } \xi \to 0, \\ -b^{-1} & \text{as } |\xi| \to \infty, \end{cases} \qquad \lambda_+ \sim \begin{cases} -a^{-1}|\xi|^2 & \text{as } \xi \to 0, \\ -b|\xi|^2 & \text{as } |\xi| \to \infty. \end{cases}$$

In particular, the solution can still be "approximated" by $e^{-at}w_0 + e^{b^{-1}t\Delta}v_0$, in a neighborhood of $\xi = 0$, and by $e^{-a^{-1}t}w_0 + e^{bt\Delta}v_0$, for large $|\xi|$, for some w_0, v_0, depending on u_0, u_1. However, some extra attention is necessary to deal with the $L^1 - L^1$ estimates, which can be harder to obtain (see Remark 2.3).

Remark 2.7. As we claimed in the beginning of this Section, linear estimates in Propositions 2.1, 2.2 and 2.4 may be trivially proved for initial data localized at intermediate frequencies, namely, away from $\xi \to 0$ and $|\xi| \to \infty$. For instance, let us prove (2.3), assuming $\hat{u}_0, \hat{u}_1$ to be supported in the annulus $A_\varepsilon = \{1 - \varepsilon \leq |\xi| \leq 1 + \varepsilon\}$, for some $\varepsilon \in (0, 1)$. In A_ε, it holds

$$|\hat{u}(t,\xi)| \leq C_\varepsilon \big(|\hat{u}_0| + t|\hat{u}_1|\big)\, e^{-t(1-\varepsilon)^2},$$

due to

$$\begin{aligned} &\hat{u} = \frac{e^{-t|\xi|^2} - |\xi|^2 e^{-t}}{1 - |\xi|^2}\, \hat{u}_0 + \frac{e^{-t|\xi|^2} - e^{-t}}{1 - |\xi|^2}\, \hat{u}_1 \qquad \text{for } |\xi| \neq 1, \\ &\hat{u} = \hat{u}_0 + t\,\hat{u}_1 \qquad \text{for } |\xi| = 1. \end{aligned}$$

In particular, as it is well known, the singularity $(1 - |\xi|^2)^{-1}$ is compensated by the difference of the two exponential terms, as $|\xi| \to 1$. By Plancherel's theorem,

$$\begin{aligned} \|\Delta u(t,\cdot)\|_{L^2} &\lesssim \||\xi|^2 \hat{u}(t,\cdot)\|_{L^2} \leq (1+\varepsilon)^2 \,\|\hat{u}(t,\cdot)\|_{L^2} \\ &\lesssim \big(\|\hat{u}_0\|_{L^2} + t\|\hat{u}_1\|_{L^2}\big)\, e^{-t(1-\varepsilon)^2}, \end{aligned}$$

so that (2.3) trivially follows.

3. Proof of Theorems 1.1, 1.2 and 1.3

The proofs of Theorems 1.1, 1.2 and 1.3 rely on a classical contraction argument.

We may write the (global) solution to the linear Cauchy problem (1.1) in the form

$$u^{\mathrm{lin}} \doteq J_0(t,x) *_{(x)} u_0(x) + J_1(t,x) *_{(x)} u_1(x)\,.$$

By Duhamel's principle, a function $u \in X$, where X is a suitable space, is a solution to (1.2), (1.10) or (1.11) if, and only if, it satisfies the equality

$$u(t,x) = u^{\mathrm{lin}}(t,x) + \int_0^t J_1(t-s,x) *_{(x)} f(u,u_t,\nabla u)(s,x)\,ds\,, \qquad \text{in } X, \tag{3.1}$$

where $f = |u|^p$, $f = |u_t|^p$ or $f = |\nabla u|^p$. To prove Theorems 1.1 and 1.2, we define the solution space

$$X \doteq \mathcal{C}\big([0,\infty), L^1 \cap H^1\big) \cap \mathcal{C}^1\big([0,\infty), L^1 \cap L^2\big), \tag{3.2}$$

with norm given by

$$\begin{aligned} \|u\|_X \doteq \sup_{t\in[0,\infty)} \Big\{ & (1+t)^{\frac{n}{4}} \|u(t,\cdot)\|_{L^2} + (1+t)^{\frac{n}{4}+\frac{1}{2}} \|\nabla u(t,\cdot)\|_{L^2} \\ & + (1+t)^{\frac{n}{4}+\frac{1}{2}} \|u_t(t,\cdot)\|_{L^2} + \|u(t,\cdot)\|_{L^1} + (1+t)\|u_t(t,\cdot)\|_{L^1} \Big\}. \end{aligned} \tag{3.3}$$

In particular, any function $u \in X$ satisfies estimates (1.5)-(1.6)-(1.7)-(1.8)-(1.9).

Thanks to linear estimates (2.6), (2.9), (2.12), (2.15) and (2.17), it follows that $u^{\mathrm{lin}} \in X$ and it satisfies

$$\|u^{\mathrm{lin}}\|_X \leq C\,\|(u_0,u_1)\|_{\mathcal{A}}\,. \tag{3.4}$$

We define the operator F such that, for any $u \in X$,

$$Fu(t,x) \doteq \int_0^t J_1(t-s,x) *_{(x)} f(u,u_t,\nabla u)(s,x)\,ds\,, \tag{3.5}$$

then we prove the estimates

$$\|Fu\|_X \leq C\|u\|_X^p\,, \tag{3.6}$$

$$\|Fu - Fv\|_X \leq C\|u-v\|_X \big(\|u\|_X^{p-1} + \|v\|_X^{p-1}\big)\,. \tag{3.7}$$

By standard arguments, since u^{lin} satisfies (3.4) and $p > 1$, from (3.6) it follows that $F + u^{\mathrm{lin}}$ maps balls of X into balls of X, for small data in $\mathcal{A}$, and estimates (3.6)-(3.7) lead to the existence of a unique solution to (3.1), that is, $u = u^{\mathrm{lin}} + Fu$, satisfying (3.4). We simultaneously gain a local and a global existence result.

Therefore, we shall only prove (3.6) and (3.7). For the sake of brevity, we will omit the proof of (3.7), which is analogous to the proof of (3.6).

We notice that, for any $u \in X$, it holds:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}\,\|u\|_X, \tag{3.8}$$

for any $q \in [1,\infty]$ if $n=1$, for any $q \in [1,\infty)$ if $n=2$, and for any $q \in [1, 2n/(n-2)]$, if $n \geq 3$, and

$$\|u_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-1} \|u\|_X, \tag{3.9}$$

for any $q \in [1,2]$. Indeed, (3.8) and (3.9) hold for $q=1,2$, as a consequence of (3.3), and so they hold for any $q \in (1,2)$, by interpolation. Moreover, since (3.3) implies

$$\|\nabla u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}-\frac{1}{2}} \|u\|_X,$$

we may use Gagliardo-Nirenberg inequality to get (3.8), for any $q \in (2,\infty]$ if $n=1$, for any $q \in (2,\infty)$ if $n=2$, and for any $q \in (2, 2n/(n-2)]$, if $n \geq 3$.

In the following, we write $p \leq n/(n-2)_+$ to mean that the finite exponent p verifies $p \leq 1 + 2/(n-2)$ if $n \geq 3$, i.e., that H^1 embeds in L^{2p}.

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $f = |u|^p$. We first prove that

$$\|\partial_t(Fu)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-1} \|u\|_X^p. \tag{3.10}$$

We use estimate (2.6) in $[0, t/2]$, and estimate (2.5) in $[t/2, t]$. Then

$$\begin{aligned}
\|\partial_t(Fu)(t,\cdot)\|_{L^2} &\leq \int_0^t \|\partial_t J_1(t-\tau,\cdot) *_{(x)} |u(\tau,\cdot)|^p\|_{L^2}\, d\tau \\
&\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-1} \left(\||u(\tau,\cdot)|^p\|_{L^1} + \||u(\tau,\cdot)|^p\|_{L^2}\right) d\tau \\
&\quad + \int_{t/2}^t (1+t-\tau)^{-1} \||u(\tau,\cdot)|^p\|_{L^2}\, d\tau \\
&\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-1} \left(\|u(\tau,\cdot)\|_{L^p}^p + \|u(\tau,\cdot)\|_{L^{2p}}^p\right) d\tau \\
&\quad + \int_{t/2}^t (1+t-\tau)^{-1} \|u(\tau,\cdot)\|_{L^{2p}}^p\, d\tau.
\end{aligned}$$

Recalling that $p \leq n/(n-2)_+$, we may use $u \in X$ and (3.8), to get:

$$\begin{aligned}
\|u(\tau,\cdot)\|_{L^p}^p &\lesssim (1+\tau)^{-\frac{n}{2}(p-1)} \|u\|_X^p, \\
\|u(\tau,\cdot)\|_{L^{2p}}^p &\lesssim (1+\tau)^{-\frac{n}{4}-\frac{n}{2}(p-1)} \|u\|_X^p.
\end{aligned}$$

Here and in the following, we use:

$$\begin{aligned}
1+t-\tau \approx 1+t, &\qquad \text{for any } \tau \in [0, t/2], \\
1+\tau \approx 1+t, &\qquad \text{for any } \tau \in [t/2, t].
\end{aligned}$$

Therefore, we obtain:

$$\begin{aligned}
\|\partial_t (Fu)(t,\cdot)\|_{L^2} &\lesssim \|u\|_X^p \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-1}\,(1+\tau)^{-\frac{n}{2}(p-1)}\,d\tau \\
&\quad + \|u\|_X^p \int_{t/2}^t (1+t-\tau)^{-1}\,(1+\tau)^{-\frac{n}{4}-\frac{n}{2}(p-1)}\,d\tau \\
&\approx \|u\|_X^p\,(1+t)^{-\frac{n}{4}-1} \int_0^{t/2} (1+\tau)^{-\frac{n}{2}(p-1)}\,d\tau \\
&\quad + \|u\|_X^p\,(1+t)^{-\frac{n}{4}-\frac{n}{2}(p-1)} \int_{t/2}^t (1+t-\tau)^{-1}\,d\tau \\
&\approx (1+t)^{-\frac{n}{4}-1}\,\|u\|_X^p,
\end{aligned}$$

thanks to $n(p-1)/2>1$, i.e., $p>1+2/n$.

Similarly, we may derive the estimates

$$\|\nabla (Fu)(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\|u\|_X^p, \tag{3.11}$$

$$\|(Fu)(t,\cdot)\|_{L^2} \lesssim \|u\|_X^p. \tag{3.12}$$

Using estimate (2.9) or, respectively, estimate (2.12), in $[0,t/2]$, and estimate (2.8) or, respectively, estimate (2.11), in $[t/2,t]$, with $m=2$, we get

$$\begin{aligned}
\|\nabla^j (Fu)(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{j}{2}}\,\big(\|u(\tau,\cdot)\|_{L^p}^p + \|u(\tau,\cdot)\|_{L^{2p}}^p\big)\,d\tau \\
&\quad + \int_{t/2}^t (1+t-\tau)^{-\frac{j}{2}}\,\|u(\tau,\cdot)\|_{L^{2p}}^p\,d\tau \\
&\lesssim \|u\|_X^p \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-\frac{j}{2}}\,(1+\tau)^{-\frac{n}{2}(p-1)}\,d\tau \\
&\quad + \|u\|_X^p \int_{t/2}^t (1+t-\tau)^{-\frac{j}{2}}\,(1+\tau)^{-\frac{n}{4}-\frac{n}{2}(p-1)}\,d\tau \\
&\approx \|u\|_X^p\,(1+t)^{-\frac{n}{4}-\frac{j}{2}} \int_0^{t/2} (1+\tau)^{-\frac{n}{2}(p-1)}\,d\tau \\
&\quad + \|u\|_X^p\,(1+t)^{-\frac{n}{4}-\frac{n}{2}(p-1)} \int_{t/2}^t (1+t-\tau)^{-\frac{j}{2}}\,d\tau \\
&\approx (1+t)^{-\frac{n}{4}-\frac{j}{2}}\,\|u\|_X^p,
\end{aligned}$$

for $j=0,1$, where we used again $p\leq n/(n-2)_+$, $u\in X$, (3.8), and $p>1+2/n$.

Finally, we derive the estimates

$$\|Fu(t,\cdot)\|_{L^1} \lesssim \|u\|_X^p, \tag{3.13}$$

$$\|\partial_t (Fu)(t,\cdot)\|_{L^1} \lesssim (1+t)^{-1}\|u\|_X^p. \tag{3.14}$$

We may use (2.15) and (2.17) to obtain

$$\begin{aligned}
\|\partial_t^k(Fu)(t,\cdot)\|_{L^1} &\lesssim \int_0^t (1+t-\tau)^{-k}\,\|u(\tau,\cdot)\|_{L^p}^p\,d\tau \\
&\lesssim \|u\|_X^p \int_0^t (1+t-\tau)^{-k}\,(1+\tau)^{-\frac{n}{2}(p-1)}\,d\tau \\
&\approx (1+t)^{-k}\,\|u\|_X^p,
\end{aligned}$$

for $k=0,1$, where we used again $p>1+2/n$, $u\in X$, and (3.8).

This concludes the proof of (3.6), for $f=|u|^p$, and so the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let $f=|u_t|^p$. In this case, the proof of (3.6) is more delicate, and singular linear estimates (2.20) and (2.21) play a fundamental role.

First we prove (3.10). Thanks to the assumptions $p\leq 2$, we may set $m=2/p\in[1,2)$, and use (2.21) in $[0,t/2]$ and (2.20) in $[t/2,t]$. We notice that

$$n\left(\frac{1}{m}-\frac{1}{2}\right)=\frac{n(p-1)}{2}<2, \tag{3.15}$$

if, and only if, $p<1+4/n$, so that the assumption $p<1+2/n$ is sufficient. Therefore, we get

$$\begin{aligned}
\|\partial_t(Fu)(t,\cdot)\|_{L^2} \lesssim \int_0^{t/2} &(1+t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-1} \\
&\times (t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|u_t(\tau,\cdot)\|_{L^p}^p+\|u_t(\tau,\cdot)\|_{L^2}^p\right)d\tau \\
&+\int_{t/2}^t (1+t-\tau)^{-1}\,(t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u_t(\tau,\cdot)\|_{L^2}^p\,d\tau.
\end{aligned}$$

Now, thanks to $u\in X$ and $p\leq 2$, using (3.9), we derive

$$\begin{aligned}
\|u_t(\tau,\cdot)\|_{L^p}^p &\lesssim (1+\tau)^{-\frac{n}{2}(p-1)-p}\,\|u\|_X^p, \\
\|u_t(\tau,\cdot)\|_{L^2}^p &\lesssim (1+\tau)^{-\left(\frac{n}{4}+1\right)p}\,\|u\|_X^p.
\end{aligned}$$

In the first integral, we may proceed as we did for $f=|u|^p$; the assumption $p>1$ is sufficient to get

$$\begin{aligned}
&\int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{m}\right)-1}\,(t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\left(\|u_t(\tau,\cdot)\|_{L^p}^p+\|u_t(\tau,\cdot)\|_{L^2}^p\right)d\tau \\
&\qquad\lesssim (1+t)^{-\frac{n}{4}-1}\|u\|_X^p\int_0^{t/2}(1+\tau)^{-\frac{n}{2}(p-1)-p}\,d\tau \approx (1+t)^{-\frac{n}{4}-1}\|u\|_X^p.
\end{aligned}$$

In the second integral, we should pay more attention, since we have

$$\begin{aligned}\int_{t/2}^t &(1+t-\tau)^{-1}\,(t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u_t(\tau,\cdot)\|_{L^2}^p\,d\tau\\ &\lesssim (1+t)^{-\left(\frac{n}{4}+1\right)p}\|u\|_X^p\int_{t/2}^t (1+t-\tau)^{-1}\,(t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\,d\tau.\end{aligned}$$

The difference with respect to the previous cases, is that, integrating, we *lose a power* $t^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}$, in the estimate:

$$\int_{t/2}^t (1+t-\tau)^{-1}\,(t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}\,d\tau \approx 1. \tag{3.16}$$

We remark that the singular power $(t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)}$ is integrable at $\tau=t$, due to (3.15).

However, the loss in (3.16) does not influence the final estimates, since, for any $p\geq 1$, we get

$$(1+t)^{-\left(\frac{n}{4}+1\right)p}\|u\|_X^p \leq (1+t)^{-\frac{n}{4}-1}\|u\|_X^p. \tag{3.17}$$

Now we prove (3.11). Setting $m=2/p$ as before, we have that

$$n\left(\frac{1}{m}-\frac{1}{2}\right)=\frac{n(p-1)}{2}<1,$$

where we now used the assumption $p<1+2/n$. We use (2.9) in $[0,t/2]$ and (2.8) in $[t/2,t]$, to obtain:

$$\begin{aligned}\|\nabla u(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\left(\|u_t(\tau,\cdot)\|_{L^p}^p+\|u_t(\tau,\cdot)\|_{L^2}^p\right)d\tau\\ &\quad+\int_{t/2}^t (1+t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}\|u_t(\tau,\cdot)\|_{L^2}^p\,d\tau\\ &\lesssim (1+t)^{-\frac{n}{4}-\frac{1}{2}}\,\|u\|_X^p.\end{aligned}$$

In particular, here we used

$$\int_{t/2}^t (1+t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}d\tau \approx (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{1}{2}},$$

due to (3.15), so that

$$(1+t)^{-\frac{n}{4}\,p-p}\int_{t/2}^t (1+t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}}d\tau \lesssim (1+t)^{\frac{1}{2}-\frac{n}{4}\,p-p}\leq (1+t)^{-\frac{n}{4}-\frac{1}{2}}.$$

Similarly, we may prove estimate (3.12). Using (2.12) in $[0,t/2]$ and (2.11) in $[t/2,t]$, we get:

$$\begin{aligned}\|u(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}} \left(\|u_t(\tau,\cdot)\|_{L^p}^p + \|u_t(\tau,\cdot)\|_{L^2}^p\right) d\tau \\ &\quad + \int_{t/2}^t (1+t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|u_t(\tau,\cdot)\|_{L^2}^p \, d\tau \\ &\lesssim (1+t)^{-\frac{n}{4}} \|u\|_X^p.\end{aligned}$$

To derive this latter it was sufficient again to use $1<p<1+4/n$. Indeed, if one is not interested in having a solution with $\nabla u(t,\cdot) \in L^2$, the bound of p from above may be relaxed to $p \leq 2$ if $n \leq 3$, and $p<1+4/n$, if $n \geq 4$.

Finally, we obtain (3.13)-(3.14) by applying (2.15) and (2.17), as in the case $f=|u|^p$; we obtain

$$\begin{aligned}\|\partial_t^k (Fu)(t,\cdot)\|_{L^1} &\lesssim \int_0^t (1+t-\tau)^{-k} \|u_t(\tau,\cdot)\|_{L^p}^p \, d\tau \\ &\lesssim \|u\|_X^p \int_0^t (1+t-\tau)^{-k} (1+\tau)^{-\frac{n}{2}(p-1)-p} \, d\tau \\ &\approx (1+t)^{-k} \|u\|_X^p,\end{aligned}$$

for $k=0,1$, using $p>1$. This concludes the proof of (3.6), and so the proof of Theorem 1.2. □

Remark 3.1. The compensating effect in (3.17) depends on the choice $m = 2/p$. A different choice for $m \in [1,2)$, verifying (2.10) and $mp \leq 2$, would led to:

$$\int_{t/2}^t (1+t-\tau)^{-1} (t-\tau)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)} \|u_t(\tau,\cdot)\|_{L^{mp}}^p \, d\tau \lesssim (1+t)^{-\frac{n}{2}(p-1/m)-p} \|u\|_X^p,$$

and

$$\frac{n}{2}\left(p-\frac{1}{m}\right)+p \geq \frac{n}{4}+1 \iff p \geq 1+\frac{n}{n+2}\left(\frac{1}{m}-\frac{1}{2}\right).$$

In particular, one cannot obtain the desired estimate, by fixing an exponent m, uniformly chosen for any p close to 1. This delicate situation shows how to grasp the critical exponent, one has to use the "correct" linear estimate, when it is necessary to compensate possible losses due to the integration of Duhamel's part of the solution.

To prove Theorem 1.3, we may follow the proof of Theorem 1.1, but now we use the solution space

$$X \doteq \mathcal{C}\big([0,\infty), W^{1,1} \cap H^2\big) \cap \mathcal{C}^1\big([0,\infty), L^1 \cap L^2\big), \tag{3.18}$$

with norm given by

$$\|u\|_X \doteq \sup_{t\in[0,\infty)} \Big\{(1+t)^{\frac{n}{4}}\Big(\|u(t,\cdot)\|_{L^2}+(1+t)\|(\Delta u,u_t)(t,\cdot)\|_{L^2}\Big)\\ +\|u(t,\cdot)\|_{L^1}+(1+t)^{\frac12}\|\nabla u(t,\cdot)\|_{L^1}+(1+t)\|u_t(t,\cdot)\|_{L^1}\Big\}. \tag{3.19}$$

As we did to prove Theorems 1.1 and 1.2, we will only prove (3.6), where now X is given by (3.18)-(3.19).

For any $u\in X$, it holds:

$$\|\nabla u(t,\cdot)\|_{L^q}\lesssim (1+t)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac12}\,\|u\|_X, \tag{3.20}$$

for any $q\in[1,\infty]$ if $n=1$, for any $q\in[1,\infty)$ if $n=2$, and for any $q\in[1,2n/(n-2)]$, if $n\geq 3$. Indeed, on the one hand, (3.20) holds for $q=1$, as a consequence of (3.19). On the other hand, since (3.19) implies

$$\|u(t,\cdot)\|_{\dot H^2}\lesssim (1+t)^{-\frac{n}{4}-1}\,\|u\|_X,$$

by the equivalence of the norm of $\|\Delta f\|_{L^2}$ and $\|f\|_{\dot H^2}$, we may use Gagliardo-Nirenberg inequality to get (3.8), for any $q\in(1,\infty]$ if $n=1$, for any $q\in(1,\infty)$ if $n=2$, and for any $q\in(1,2n/(n-2)]$, if $n\geq 3$.

Proof of Theorem 1.3. We first prove

$$\|\Delta(Fu)(t,\cdot)\|_{L^2}+\|\partial_t(Fu)(t,\cdot)\|_{L^2}\lesssim (1+t)^{-1}\|u\|_X^p, \tag{3.21}$$

as we did to prove (3.10), in the case $f=|u|^p$. We use estimate (2.4) in $[0,t/2]$, and estimate (2.3) in $[t/2,t]$. Then

$$\begin{aligned}\|\Delta(Fu)(t,\cdot)\|_{L^2}&+\|\partial_t(Fu)(t,\cdot)\|_{L^2}\\ &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{4}-1}\left(\|\nabla u(\tau,\cdot)\|_{L^p}^p+\|\nabla u(\tau,\cdot)\|_{L^{2p}}^p\right)d\tau\\ &\quad+\int_{t/2}^t(1+t-\tau)^{-1}\,\|\nabla u(\tau,\cdot)\|_{L^{2p}}^p\,d\tau.\end{aligned}$$

Now, using $u\in X$, (3.20), and $p\leq n/(n-2)_+$, we get:

$$\|\nabla u(\tau,\cdot)\|_{L^p}^p\lesssim (1+\tau)^{-\frac{n}{2}(p-1)-\frac{p}{2}}\,\|u\|_X^p,$$
$$\|\nabla u(\tau,\cdot)\|_{L^{2p}}^p\lesssim (1+\tau)^{-\frac{n}{4}-\frac{n}{2}(p-1)-\frac{p}{2}}\,\|u\|_X^p.$$

Therefore, we obtain:

$$\begin{aligned}
&\|\Delta(Fu)(t,\cdot)\|_{L^2} + \|\partial_t(Fu)(t,\cdot)\|_{L^2} \\
&\quad\lesssim \|u\|_X^p \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}-1} (1+\tau)^{-\frac{n}{2}(p-1)-\frac{p}{2}}\, d\tau \\
&\qquad + \|u\|_X^p \int_{t/2}^t (1+t-\tau)^{-1} (1+\tau)^{-\frac{n}{4}-\frac{n}{2}(p-1)-\frac{p}{2}}\, d\tau \\
&\quad\approx \|u\|_X^p (1+t)^{-\frac{n}{4}-1} \int_0^{t/2} (1+\tau)^{-\frac{n}{2}(p-1)-\frac{p}{2}}\, d\tau \\
&\qquad + \|u\|_X^p (1+t)^{-\frac{n}{4}-\frac{n}{2}(p-1)-\frac{p}{2}} \int_{t/2}^t (1+t-\tau)^{-1}\, d\tau \\
&\quad\approx (1+t)^{-\frac{n}{4}-1} \|u\|_X^p,
\end{aligned}$$

thanks to $p > 1 + 1/(n+1)$. Let us prove (3.12). Using estimate (2.12), in $[0, t/2]$, and estimate (2.11), in $[t/2, t]$, with $m = 2$, we get

$$\begin{aligned}
\|(Fu)(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}} \left(\|\nabla u(\tau,\cdot)\|_{L^p}^p + \|\nabla u(\tau,\cdot)\|_{L^{2p}}^p\right) d\tau \\
&\quad + \int_{t/2}^t \|\nabla u(\tau,\cdot)\|_{L^{2p}}^p\, d\tau \\
&\lesssim \|u\|_X^p \int_0^{t/2} (1+t-\tau)^{-\frac{n}{4}} (1+\tau)^{-\frac{n}{2}(p-1)-\frac{p}{2}}\, d\tau \\
&\quad + \|u\|_X^p \int_{t/2}^t (1+\tau)^{-\frac{n}{4}-\frac{n}{2}(p-1)-\frac{p}{2}}\, d\tau \\
&\approx \|u\|_X^p (1+t)^{-\frac{n}{4}} \int_0^{t/2} (1+\tau)^{-\frac{n}{2}(p-1)-\frac{p}{2}}\, d\tau \\
&\quad + \|u\|_X^p (1+t)^{-\frac{n}{4}-\frac{n}{2}(p-1)-\frac{p}{2}} \int_{t/2}^t 1\, d\tau \\
&\approx (1+t)^{-\frac{n}{4}} \|u\|_X^p,
\end{aligned}$$

where we used again $u \in X$, (3.20), $p \leq n/(n-2)_+$ and $p > 1 + 1/(n+1)$.

Finally, we derive estimates (3.13), (3.14), and

$$\|\nabla(Fu)(t,\cdot)\|_{L^1} \lesssim (1+t)^{-\frac{1}{2}} \|u\|_X^p. \tag{3.22}$$

We may use (2.15), (2.16), and (2.17), to estimate

$$\begin{aligned}
\|\nabla^j \partial_t^k (Fu)(t,\cdot)\|_{L^1} &\lesssim \int_0^t (1+t-\tau)^{-\frac{j}{2}-k} \|\nabla u(\tau,\cdot)\|_{L^p}^p\, d\tau \\
&\lesssim \|u\|_X^p \int_0^t (1+t-\tau)^{-\frac{j}{2}-k} (1+\tau)^{-\frac{n}{2}(p-1)-\frac{p}{2}}\, d\tau \\
&\lesssim (1+t)^{-\frac{j}{2}-k} \|u\|_X^p,
\end{aligned}$$

for $j + k = 0, 1$, where we used again $p > 1 + 1/(n+1)$, $u \in X$, and (3.20).

This concludes the proof of (3.6), for $f = |\nabla u|^p$, and so the proof of Theorem 1.3. $\square$

Appendix A. L^1 multipliers estimates

In this appendix we prove an L^1 estimate for multipliers, localized at low and high frequencies. The employment of these estimates allow us to derive $L^1 - L^1$ estimates for the solution to (1.1). The technique employed is well-known, but we give some details for the ease of reading.

Lemma A.1. *Let $n \geq 1$ and χ_0 be a $\mathcal{C}^\infty$ function, supported in $B_1(0) = \{|\xi| < 1\}$, and constant in some neighborhood of the origin. Then:*

$$K_2 = \mathfrak{F}^{-1}\left(\frac{|\xi|^2}{1-|\xi|^2}\,\chi_0\right) \in L^1, \qquad K_1 = \mathfrak{F}^{-1}\left(\frac{\xi}{1-|\xi|^2}\,\chi_0\right) \in L^1. \quad \text{(A.1)}$$

Proof. Let $a \in (0,1)$ be such that $\operatorname{supp}\chi_0 \subset B_a(0) = \{|\xi| < a\}$, and let $g_2(\xi) = |\xi|^2$ and $g_1(\xi) = \xi$. Then

$$K_j(x) = (2\pi)^{-n} \int_{|\xi|\leq a} e^{ix\xi}\, \frac{g_j(\xi)}{1-|\xi|^2}\,\chi_0(\xi)\,d\xi.$$

It is clear that $K_j \in L^\infty$, in particular, it is in L^1_{loc}. Let $|x| \geq a^{-1}$. Thanks to

$$e^{ix\xi} = \sum_{j=1}^n \frac{-ix_j}{|x|^2}\,\partial_{\xi_j} e^{ix\xi}, \quad \text{(A.2)}$$

after integrating by parts $n-1$ times (the boundary terms vanish, since χ_0 identically vanishes near $\{|\xi| = a\}$), we obtain:

$$K_j(x) = |x|^{-(n-1)} \sum_{|\gamma|=n-1} c_{n,\gamma} \left(\frac{ix}{|x|}\right)^\gamma \int_{|\xi|\leq a} e^{ix\xi}\,\partial_\xi^\gamma \left(\frac{g_j(\xi)}{1-|\xi|^2}\,\chi_0(\xi)\right) d\xi.$$

Indeed,

$$\begin{aligned} K_j(x) &= (2\pi)^{-n} \sum_{\ell=1}^n \frac{ix_\ell}{|x|^2} \int_{|\xi|\leq a} e^{ix\xi}\,\partial_{\xi_\ell} \left(\frac{g_j(\xi)}{1-|\xi|^2}\,\chi_0(\xi)\right) d\xi \\ &= (2\pi)^{-n} \sum_{\ell=1}^n \sum_{k=1}^n \frac{-x_\ell x_k}{|x|^4} \int_{|\xi|\leq a} e^{ix\xi}\,\partial_{\xi_k}\partial_{\xi_\ell} \left(\frac{g_j(\xi)}{1-|\xi|^2}\,\chi_0(\xi)\right) d\xi \\ &= \dots \end{aligned}$$

We split each of the integral in K_j into two parts. We immediately get

$$\int_{|\xi|\leq|x|^{-1}} \left|\partial_\xi^\gamma \left(\frac{g_j(\xi)}{1-|\xi|^2}\,\chi_0(\xi)\right)\right| d\xi \lesssim \int_{|\xi|\leq|x|^{-1}} |\xi|^{j-(n-1)}\,d\xi \lesssim |x|^{-1-j}.$$

Integrating one more time the remaining integral, we obtain:

$$\int_{|x|^{-1}\leq|\xi|\leq a} e^{ix\xi}\,\partial_\xi^\gamma \left(\frac{g_j(\xi)}{1-|\xi|^2}\,\chi_0(\xi)\right)\,d\xi = \sum_{j=1}^n \frac{-ix_j}{|x|^2}\int_{|\xi|=|x|^{-1}} e^{ix\xi}\,\partial_\xi^\gamma \left(\frac{g_j(\xi)}{1-|\xi|^2}\,\chi_0(\xi)\right)\,dS + \sum_{j=1}^n \frac{ix_j}{|x|^2}\int_{|x|^{-1}\leq|\xi|\leq a} e^{ix\xi}\,\partial_{\xi_j}\partial_\xi^\gamma \left(\frac{g_j(\xi)}{1-|\xi|^2}\,\chi_0(\xi)\right)\,d\xi.$$

The first term in the right-hand side is bounded by $|x|^{-1-j}$, whereas we may perform one more step of integration on the second one, which leads us to estimate

$$\int_{|x|^{-1}\leq|\xi|\leq a} |\xi|^{j-(n+1)}\,d\xi \lesssim \begin{cases} 1 & \text{if } j=2, \\ 1+\log|x| & \text{if } j=1. \end{cases}$$

In turns, we obtain: $|K_1(x)|\lesssim|x|^{-(n+1)}(1+\log|x|)$, and $|K_2(x)|\lesssim|x|^{-(n+1)}$, for large $|x|$. That is, $K_j\in L^1$. □

Lemma A.2. *Let $n\geq 1$ and χ_1 be a $\mathcal{C}^\infty$ function, supported in $\mathbb{R}^n\setminus\bar{B}_1(0)=\{|\xi|>1\}$, and constant for $|x|\geq R$, for some $R>1$. Then:*

$$K_0=\mathfrak{F}^{-1}\left(\frac{1}{|\xi|^2-1}\,\chi_1\right)\in L^1, \qquad K_1=\mathfrak{F}^{-1}\left(\frac{\xi}{|\xi|^2-1}\,\chi_1\right)\in L^1. \tag{A.3}$$

Proof. Let $b>1$, be such that $\operatorname{supp}\chi_1\subset\mathbb{R}^n\setminus\bar{B}_b(0)=\{|\xi|>b\}$. Recalling (A.2), and following the proof of Lemma A.1, we integrate by parts $n-1$ times:

$$K_j(x)=|x|^{-(n-1)}\,(2\pi)^{-n}\sum_{|\gamma|=n-1}\left(\frac{ix}{|x|}\right)^\gamma\int_{|\xi|\geq a} e^{ix\xi}\,\partial_\xi^\gamma\left(\frac{g_j(\xi)}{|\xi|^2-1}\,\chi_1(\xi)\right)\,d\xi, \tag{A.4}$$

with $g_0(\xi)=1$ and $g_1(\xi)=\xi$. We immediately obtain

$$|K_0(x)|\lesssim|x|^{-(n-1)}\int_{|\xi|\geq a}|\xi|^{-2-(n-1)}\,d\xi\lesssim|x|^{-(n-1)},$$

whereas, to treat K_1 for small $|x|$, we split (A.4) into two integrals. On the one hand, we easily obtain:

$$|x|^{-(n-1)}\int_{a\leq|\xi|\leq|x|^{-1}}|\xi|^{-1-(n-1)}\,d\xi\lesssim|x|^{-(n-1)}(1+|\log|x||).$$

On the other hand, performing one more step of integration by parts of the remaining integral, we get

$$\int_{|x|^{-1}\leq|\xi|} e^{ix\xi}\,\partial_\xi^\gamma\left(\frac{\xi}{|\xi|^2-1}\,\chi_1(\xi)\right)d\xi = \sum_{j=1}^n \frac{-ix_j}{|x|^2}\int_{|\xi|=|x|^{-1}} e^{ix\xi}\,\partial_\xi^\gamma\left(\frac{\xi}{|\xi|^2-1}\,\chi_1(\xi)\right)dS + \sum_{j=1}^n \frac{ix_j}{|x|^2}\int_{|x|^{-1}\leq|\xi|} e^{ix\xi}\,\partial_{\xi_j}\partial_\xi^\gamma\left(\frac{\xi}{|\xi|^2-1}\,\chi_1(\xi)\right)d\xi,$$

which we may control by $|x|^{-(n-1)}$. To estimate K_j, $j=0,1$, for large $|x|$, it is sufficient to perform two more steps of integration by parts in (A.4), obtaining:

$$|K_j(x)| \lesssim |x|^{-(n+1)}\int_{|\xi|\geq a}|\xi|^{j-(n+3)}\,d\xi \lesssim |x|^{-(n+1)}.$$

Summarizing, we proved that $K_0, K_1 \in L^1$ (and also in L^q, for any $q < n/(n-1)$). □

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