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ALGEBRAIC CONSTRUCTION AND NUMERICAL BEHAVIOR OF A NEW S-CONSISTENT DIFFERENCE SCHEME FOR THE 2D NAVIER-STOKES EQUATIONS

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Abstract. In this paper we consider a regular grid with equal spatial spacings and construct a new finite difference approximation (difference scheme) for the system of two-dimensional Navier-Stokes equations describing the unsteady motion of an incompressible viscous liquid of constant viscosity. In so doing, we use earlier constructed discretization of the system of three equations: the continuity equation and the proper Navier-Stokes equations. Then, we compute the canonical Gröbner basis form for the obtained discrete system. It gives one more difference equation which is equivalent to the pressure Poisson equation modulo difference ideal generated by the Navier-Stokes equations, and thereby comprises a new finite difference approximation (scheme). We show that the new scheme is strongly consistent. Besides, our computational experiments demonstrate much better numerical behaviour of the new scheme in comparison with the other strongly consistent schemes we constructed earlier and with the scheme which is not strongly consistent.

1. Introduction

Numerical solving of partial differential equations (PDE) is a fundamental task of applied mathematics and engineering. There are three numerical methods which have been used extensively for solving of PDE: the finite element method, the finite volume method and the finite difference method. In the present paper we consider the last method described in a rather large number of textbooks (see, for examle, [15, 21, 23, 24]). Its application is based on a finite difference approximation (FDA) to a PDE.

The standard way to derive a FDA resides in the approximation of partial derivatives by linear combinations of function values at the grid points. In the case of a single differential equation, such an approximation must provide *consistency* (cf. [15], p.15; [21], p.25; [23], Sect.8.4) of the obtained difference equation with the differential one. If one deals with a PDE system, then its FDA, such that every difference equation in the discretized system, is consistent with the corresponding

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differential equation in the PDE system is called *equation-wise consistent* [8] or *weakly consistent (w-consistent)* [4]. In doing so, if one rewrites the FDA into a fully equivalent (i.e. preserving the solution set) form, then it may happen that the difference equations in the rewritten FDA in the continuous limit (i.e. in the limit when the grid spacings go to zero) give a PDE system whose solution set is not equal to the solution set of the original differential system. If a FDA to a given PDE system is such that any equivalent form of the FDA in the continuous limit gives an equivalent form of the PDE system, then the FDA is called *strongly consistent* or *s-consistent* (cf. [4, 8]). Given a polynomially nonlinear PDE system and its FDA on a regular grid, one can verify s-consistency of the FDA by its transformation into a Gröbner basis form [4].

If a FDA inherits at the discrete level all fundamental algebraic properties (e.g. conservation local laws) of the PDE system under consideration, then the FDA is a *mimetic* or *compatible* discretization (see, for example, book [3], its bibliography and articles [19, 20]). Such FDA is s-consistent. While mimetic methods initially construct a discrete mathematical analog of a physical conservation or constitutive law (cf. [3], Ch.1, p.2), for an s-consistent FDA the numerical scheme for such conservation law (cf. [24], Ch.9) is a difference–algebraic consequence of the FDA (see Definition 2.10). Besides, s-consistency is expected to be necessary for *convergence* of the FDA as a difference scheme, since it has been adopted that the *convergence* is provided if a given FDA to the PDE is consistent and *stable*. This adaptation extends of the brilliant Lax equivalence theorem [21, 23] rigorously proven for the initial value problem posed for a single linear PDE ([15], Thm. 5.1, p.159; [21], Thm.10.5.1, p.262; [23], Thm.8.4.1, p.61).

In [5] three different FDAs (different schemes) for the two-dimensional Navier-Stokes equations describing the unsteady motion of an incompressible viscous liquid of constant viscosity were constructed. The method used for construction was proposed in [6]. It combines the finite volume method, numerical integration and the difference elimination of the grid functions for partial derivatives from the discrete equations obtained after numerical integration. The elimination was performed by means of difference Gröbner bases [4, 7, 12]. The s-consistency check has shown that two of the generated FDA are s-consistent, and the third one is not. According to our computational experiments done in [1], s-consistent FDAs have better numerical behavior than the FDAs which are not s-consistent.

In the given paper we derive, in addition to those produced in [5] and studied in [1], one more s-consistent FDA to the Navier-Stokes equations. For this purpose we exploit the FDA to the 2D Navier-Stokes PDE system that is comprised of the proper Navier-Stokes equations and the continuity equation, and does not include the pressure Poisson equation. Then, by making use of the algorithms which are described in [7, 12] we compute a Gröbner basis for the obtained difference system. As a result, we obtain an additional difference equation that is equivalent to the pressure Poisson equation modulo the difference ideal generated by the polynomials occurring in the Navier-Stokes equations. We prove s-consistency of the new FDA and compare its numerical behavior with the other s-consistent FDA constructed in [5] and also with the FDA constructed in [1] which is not s-consistent. As benchmarks we use the following two exact solutions to the Navier-Stokes equations: (i) the unsteady flow solution originally found in [22] and as a benchmark used

firstly in [14]; (ii) Kavasznay' steady flow problem [11]. The benchmarking shows a significant superiority of the new FDA over the others.
This paper is organized as follows. In Section 2 we consider the 2D Navier-Stokes equations and specify for them basic definitions and notions of differential and difference algebra used in the next sections. In Section 3 we give definition of s-consistency and outline procedure of its verification. The derivation of the new difference approximation to the Navier-Stokes equations, as a difference scheme, is considered in Section 4. In Section 5 we describe the numerical implementation of our new dicretization. Section 6 presents our numerical experiments. Concluding remarks are given in Section 7.

2. Navier-Stokes equations and related notions of differential and difference algebra

We consider the two-dimensional Navier-Stokes equations describing the unsteady motion of an incompressible viscous liquid of constant viscosity in the following dimensionless form

$$F := \begin{cases} f_1 := & u_x + v_y = 0\,, \\ f_2 := & u_t + uu_x + vu_y + p_x - \frac{1}{\mathrm{Re}}\,\Delta\, u = 0\,, \\ f_3 := & v_t + uv_x + vv_y + p_y - \frac{1}{\mathrm{Re}}\,\Delta\, v = 0\,. \end{cases} \tag{1}$$

Here (u, v) is the velocity field, p is the pressure, the constant Re is the Reynolds number, f_1 is the continuity equation, f_2 and f_3 are the proper Navier-Stokes equations.

A *differential polynomial* associated with system (1) is a polynomial in the independent variables u, v, p and their partial derivatives w.r.t. x, y, t with coefficients belonging to $\mathbb{Q}(\mathrm{Re})$, the field of rational functions in Re with rational coefficients. The set of all possible differential polynomials, including zero one, closed under operations of addition, multiplication and action of partial derivatives $\partial_t, \partial_x, \partial_y$, forms the *differential polynomial ring*. We shall denote this ring by

$$\mathcal{R} := \mathbb{Q}(\mathrm{Re})[u, v, p]\,.$$

Note that the left-hand sides f_1, f_2, f_3 of the Navier-Stokes equations are also differential polynomials and elements in this ring.

Definition 2.1. [4] *A* (differential–algebraic) consequence *of* (1) *is a PDE* $f = 0$ *where a differential polynomial* $f \in \mathcal{R}$ *vanishes on each solution to* (1)[1].

An important consequence of system (1) is given by $f_4 = 0$ where

$$f_4 := (f_1)_t - (f_2)_x = \,\Delta\, p + u_x^2 + 2v_x u_y + v_y^2\,, \tag{2}$$

This equation is the well-known pressure Poisson equation [9].

Definition 2.2. *The* differential ideal *generated by polynomial set* $F := \{f_1, f_2, f_3\}$ *and denoted by* $\mathcal{I} := [F]$ *is the smallest subset of* $\mathcal{R}$ *containing* F *and satisfying*

$$(\forall \delta \in \{\,\partial_t, \partial_x, \partial_y\,\})\ (\forall a, b \in \mathcal{I}\,)\ (\forall c \in \mathcal{R}\,)\quad [\,a + b \in \mathcal{I}\,,\ a\,c \in \mathcal{I},\ \delta(a) \in \mathcal{I}\,]\,.$$

[1] We consider solutions (cf. [17], p.97) which are analytic in an open and connected domain of $\mathbb{C}^3$ with coordinates t, x, y.

Definition 2.3. *Let $\mathcal{P}$ be the set of partial derivatives:*

$$\mathcal{P} := \{\, \partial_t^n \partial_x^j \partial_y^k w \mid n,j,k \in \mathbb{N}_{\geq 0},\ w \in \{u,v,p\}\,\}\,.$$

A total ordering $\succ$ on $\mathcal{P}$ is ranking *if for any $q, r \in \mathcal{P}$ such that $q \succ r$ the relations*

$$\delta(q) \succ \delta(r)\,, \quad \delta(q) \succ r\,,$$

hold for all $\delta \in \{\, \partial_t, \partial_x, \partial_y \,\}$. Given a ranking, every non-constant polynomial $f \in \mathcal{R}$ has the highest ranking derivative occurring in f. This derivative is the leader *of polynomial f. If the functions u, v, p are compared first and the elements in the monoid of derivations $\{\partial_t^n \partial_x^j \partial_y^k \mid n,j,k \in \mathbb{N}_{\geq 0}\}$ second, then the ranking is called* elimination. *Otherwise, the ranking is called* orderly.

If one chooses the elimination ranking $\succ$ on partial derivatives compatible with $p \succ u \succ v$ and $\partial_t \succ \partial_x \succ \partial_y$ and such that

$$u_t \succ v_t \succ p_x \succ p_y \succ u_x \succ u_y \succ v_x \succ v_y\,, \tag{3}$$

then the consequence (2) of (1) is the only *integrability condition*[2]. The inclusion of (2) into (1) makes the system *involutive* [5]

$$\begin{cases} f_1 = & \underline{u_x} + v_y = 0\,, \\ f_2 = & \underline{u_t} + uu_x + vu_y + p_x - \frac{1}{\mathrm{Re}}\,\Delta\, u = 0\,, \\ f_3 = & \underline{v_t} + uv_x + vv_y + p_y - \frac{1}{\mathrm{Re}}\,\Delta\, v = 0\,, \\ f_4 = & \underline{p_{xx}} + p_{yy} + u_x^2 + 2v_x u_y + v_y^2 = 0\,. \end{cases} \tag{4}$$

The underlined terms in (4) are leaders.

The completion algorithm based on differential Thomas decomposition (cf. [2], Sect.3; [17], Sect.2.2) for the input f_1, f_2, f_3 outputs the slightly different involutive form of (1) with the same leaders as in (5)

$$\begin{cases} u_x + v_y = 0\,, \\ \frac{1}{\mathrm{Re}}(u_{yy} - v_{xy} - uv_y) - vu_y - u_t - p_x = 0\,, \\ \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}) - uv_x - vv_y - v_t - p_y = 0\,, \\ 2v_x u_y + \Delta\, p + 2v_y^2 = 0\,. \end{cases} \tag{5}$$

The involutive system (5) is obtained from (4) by the Gröbner (see [16], Def.7) or Janet (cf. [2], Alg.3.3; [17], Alg.2.2.40) (inter)reduction under ranking (3). We prefer to use (4) since it preserves the symmetry ($u \Leftrightarrow v, x \Leftrightarrow y$) of the initial equations (1) and helps to keep this symmetry at the discrete level.

Each of the involutive systems (4) and (5) can be used to check whether a given differential polynomial $f \in \mathcal{R}$ is a consequence of (1). The differential ideal $[F] = [F \cup \{f_4\}]$ in Definition 2.2 is *radical*, since the ideal generated by polynomials in any PDE system outputted by the Thomas decomposition algorithm (see [2], Sect.3; [17], Sect.2.2) is radical, what means

$$p^i \in \mathcal{I}\ \ (i \in \mathbb{N}_{\geq 1}) \Longrightarrow p \in \mathcal{I}\,.$$

Therefore, f is a consequence of (1) if and only if $f \in \mathcal{I}$ (cf. [10], p.6). Both (4) and (5) are *differential Gröbner bases* of $\mathcal{I}$. It can be readily verified by hand with

[2]For definition of integrability conditions, and for related algebraic and geometric aspects of completion of differential systems to involution we refer to [18], in particular, Sect.2.3 and Sect.7.2.

the Olivier algorithm (see [16], p.314). Therefore, the reduction of f modulo (4) or (5) is zero if and only if $f \in \mathcal{I}$.

It is easy to rewrite the equations in system (4) as conservation laws [24]

$$
\left\{
\begin{aligned}
f_1 = &\quad \operatorname{div}(u,\, v) = 0\,, \\
f_2 = &\quad u_t + \operatorname{div}\left(u^2 + p - \tfrac{1}{\mathrm{Re}} u_x,\, vu - \tfrac{1}{\mathrm{Re}} u_y\right) = 0\,, \\
f_3 = &\quad v_t + \operatorname{div}\left(uv - \tfrac{1}{\mathrm{Re}} v_x,\, v^2 + p - \tfrac{1}{\mathrm{Re}} v_y\right) = 0\,,
\end{aligned}
\right. \tag{6}
$$

where $\operatorname{div}(a, b) := a_x + b_y$.

Now we consider a regular grid in the space $\mathbb{R}^3$ of independent variables (t, x, y) with the grid spacings τ, h, h

$$
t_{n+1} - t_n = \tau > 0\,, \quad x_{j+1} - x_j = y_{k+1} - y_k = h > 0\,, \tag{7}
$$

and introduce the conventional notations (cf. [21, 23]) for the grid functions

$$
u^n_{j,k} := u(n\,\tau, j\,h, k\,h), \quad v^n_{j,k} := v(n\,\tau, j\,h, k\,h), \quad p^n_{j,k} := (n\,\tau, j\,h, k\,h)\,.
$$

Because of the parameters Re, τ, h, we consider the field $\mathbb{K} = \mathbb{Q}(\mathrm{Re}, \tau, h)$ of rational functions in Re, τ, h with rational coefficients and define the (infinitely generated) polynomial algebra

$$
\tilde{\mathcal{R}} = \mathbb{K}[u^n_{j,k}, v^n_{j,k}, p^n_{j,k} \mid n, j, k \geq 0]
$$

i.e. the (infinite) set of polynomials in $u^n_{j,k}, v^n_{j,k}, p^n_{j,k}$ with coefficients from $\mathbb{K}$ and with operations of addition and multiplication[3].

We endow $\tilde{\mathcal{R}}$ with the partial shift operators $\sigma_x, \sigma_y, \sigma_t$, that is, $\tilde{\mathcal{R}}$ is the algebra of the *difference polynomials* in the variables $u = u^0_{0,0}, v = v^0_{0,0}$ and $p = p^0_{0,0}$. In doing so, the notion of difference polynomial a perfect analogy of the above described notion of differential polynomial if one replaces the derivation operators with the shift operators. By definition, $\sigma_x : \tilde{\mathcal{R}} \to \tilde{\mathcal{R}}$ is the $\mathbb{K}$-algebra endomorphism such that

$$
u^n_{j,k} \mapsto u^n_{j+1,k}\,, \quad v^n_{j,k} \mapsto v^n_{j+1,k}\,, \quad p^n_{j,k} \mapsto p^n_{j+1,k}\,.
$$

In the same way, one defines the other shift operators σ_y and σ_t.

The following definition is a difference analogue of Definition 2.2.

Definition 2.4. *Given a finite set $\tilde{F} := \{\, \tilde{f}_1, \dots, \tilde{f}_m \,\} \subset \tilde{\mathcal{R}}$ of difference polynomials, the* difference ideal *or* $\sigma-$ideal *([13], p.104) generated by the set $\tilde{F}$ and denoted by $\tilde{\mathcal{I}} := [\tilde{F}]$ is the smallest subset of $\tilde{\mathcal{R}}$ containing $\tilde{F}$ and satisfying*

$$
(\,\forall a, b \in \tilde{\mathcal{I}}\,),\ (\,\forall c \in \tilde{\mathcal{R}}\,)\ (\,\forall \sigma \in \{\, \sigma_t, \sigma_x, \sigma_y \,\}\,) \quad [\,a + b \in \tilde{\mathcal{I}}\,, \quad a\,c \in \tilde{\mathcal{I}}\,, \quad \sigma \circ a \in \tilde{\mathcal{I}}\,].
$$

A difference polynomial $\tilde{f} \in \tilde{\mathcal{R}}$ is a finite sum of *monomials*

$$
\tilde{f} = \sum_i \alpha_i m_i\,, \quad \alpha \in \mathbb{Q}(\mathrm{Re}, \tau, h)\,, \quad m_i := (\theta_1 \circ u^0_{0,0})^{i_1} (\theta_1 \circ v^0_{0,0})^{i_2} (\theta_1 \circ p^0_{0,0})^{i_3}\,.
$$

Here $i_1, i_2, i_3 \in \mathbb{N}_{\geq 0}$ and $\theta_1, \theta_2, \theta_3 \in \Theta$ where

$$
\Theta := \{\, \sigma^n_t \circ \sigma^j_x \circ \sigma^k_y \mid i, j, k \in \mathbb{N}_{\geq 0} \,\}.
$$

In perfect analogy to the notion of differential ranking (Definition 3), we introduce a notion of difference ranking (cf.[13], p.129).

[3]Hereafter we will use the tilde mark (˜) placed over letters denoting difference polynomials and their sets.

Definition 2.5. *Let $\mathcal{D}$ be the following set:*

$$\mathcal{D} := \{\, \theta \circ w \mid \theta \in \Theta,\ w \in \{u^0_{0,0}, v^0_{0,0}, p^0_{0,0}\}\,\}\,.$$

A total ordering $\succ$ on $\mathcal{D}$ is difference ranking *if for any $q, r \in \mathcal{D}$ such that $q \succ r$ the relations*

$$\sigma(q) \succ \sigma(r)\,, \quad \sigma(q) \succ r\,,$$

hold for all $\sigma \in \{\sigma_t, \sigma_x, \sigma_y\}$. Given such a ranking, every non-constant polynomial $\tilde f \in \tilde{\mathcal{R}}$ has the highest ranking variable $\theta \circ w$ occurring in $\tilde f$. This variable is the leader *of polynomial $\tilde f$. If the functions u, v, p are compared first and the shift operators θ second, then the ranking is called* elimination. *Otherwise, the ranking is called* orderly.

Definition 2.6. *A total ordering $\succ$ on the set $\mathcal{M}$ of difference monomials*

$$\mathcal{M} := \{\, \sigma_t^n \circ \sigma_x^j \circ \sigma_y^k \circ w \mid n, j, k \in \mathbb{N}_{\geq 0},\ w \in \{u, v, p\}\,\}$$

is admissible *if it extends a ranking and satisfies*

$$(\forall\, m \in \mathcal{M} \setminus \{1\})\ [m \succ 1]\ \wedge\ (\forall\, \theta \in \Sigma)\ (\forall\, a, b, c \in \mathcal{M}\,)\ [\,a \succ b \Longleftrightarrow c\,\theta \circ a \succ c\,\theta \circ b\,].$$

As an example of admissible monomial ordering, we indicate a lexicographical ordering compatible with a lexicographical ranking. This monomial ordering is similar to the lexicographical monomial ordering used in differential algebra (Definition 2 in [16]). Given an admissible ordering $\succ$, every difference polynomial $\tilde f$ has the *leading monomial* $\mathrm{lm}(\tilde f) \in \mathcal{M}$ with the *leading coefficient* $\mathrm{lc}(\tilde f)$. In what follows every difference polynomial is assumed to be normalized by the division of the polynomial by its leading coefficient. This provides $(\forall \tilde f \in \tilde{\mathcal{R}}\,)\ [\,\mathrm{lc}(\tilde f) = 1\,]$.

If for $a, b, c \in \mathcal{M}$ the equality $b = c\,\theta \circ a$ holds for some $\theta \in \Theta$ and then we shall say that a divides b and write $a \mid b$. It is easy to see that this divisibility relation yields a partial order.

Now we can present a definition of difference Gröbner (standard) basis – a universal algorithmic tool in the difference polynomial algebra.

Definition 2.7. [4, 7, 12] *Given a σ-ideal $\tilde{\mathcal{I}}$ and an admissible monomial ordering $\succ$, a subset $\tilde G \subset \tilde{\mathcal{I}}$ is its* (difference) standard basis (cf. [16])*, if $[\tilde G] = \tilde{\mathcal{I}}$ and*

$$(\,\forall\, \tilde f \in \mathcal{I}\,)(\,\exists\, \tilde g \in \tilde G\,)\ \ [\,\mathrm{lm}(\tilde g) \mid \mathrm{lm}(\tilde f)\,]\,.$$

If the standard basis is finite it is called a Gröbner basis.

This definition is not constructive. It does not give a recipe for construction of a Gröbner basis. The following definition and theorem provide such a recipe.

Definition 2.8. [4, 7, 12] *Given an admissible ordering, and normalized difference polynomials $\tilde p, \tilde q$, the polynomial $S(\tilde p, \tilde q) := m_1\,\theta_1 \circ \tilde p - m_2\,\theta_2 \circ \tilde q$ is called* S-polynomial *associated to $\tilde p$ and $\tilde q$* [4]*, if $m_1\,\theta_1 \circ \mathrm{lm}(\tilde p) = m_2\,\theta_2 \circ \mathrm{lm}(\tilde q)$ with co-prime $m_1\,\theta_1$ and $m_2\,\theta_2$.*

Theorem 2.9. [4, 7, 12] *Given an ideal $\tilde{\mathcal{I}} \subset \tilde{\mathcal{R}}$ and an admissible ordering $\succ$, a set of polynomials $\tilde G \subset \tilde{\mathcal{I}}$ is a standard basis of $\tilde{\mathcal{I}}$, if and only if $\mathrm{NF}(S(\tilde p, \tilde q), \tilde G) = 0$ for all S-polynomials associated with the polynomials in $\tilde G$.*

[4]For $\tilde p = \tilde q$ we shall say that S-polynomial is associated with $\tilde p$.

Here $\mathrm{NF}(S(\tilde p,\tilde q),\tilde G)$ denotes the simplified (reduced) value (*normal form*) of $S(\tilde p,\tilde q)$ modulo set $\tilde G$ which is computed by a finite chain of elementary reductions. For an algorithmic construction of difference Gröbner bases we refer to [4, 7, 12].

The difference analogue of Definition 2.1 is the following one.

Definition 2.10. [4] *Given a system of difference equations*

$$\tilde f_1 = 0, \ldots, \tilde f_m = 0\,, \quad \tilde f_i \in \tilde{\mathcal{R}}\,, \quad i = 1,\ldots,m\,, \quad m \in \mathbb{N}_{\geq 1}\,, \tag{8}$$

its (difference–algebraic) consequence *is a difference equation* $\tilde f = 0$, $\tilde f \in \tilde{\mathcal{R}}$ *such that* $\tilde f$ *vanishes on each solution to the system* (8).

However, as distinct from the differential case, the ideal generated by polynomials $\tilde f \in \tilde{\mathcal{R}}$ which vanish on all common solutions to (8)[5] is not the radical difference ideal, but the *perfect difference ideal* [13, 25] defined as follows.

Definition 2.11. ([13], Def.2.3.1) *A* perfect difference ideal *generated by a set* $\tilde F \in \tilde{\mathcal{R}}$ *and denoted by* $[\![\tilde F]\!]$ *is the smallest difference ideal containing* $\tilde F$ *and such that for any* $\tilde f \in \tilde{\mathcal{R}}$, $\theta_1,\ldots,\theta_r \in \Theta$ *and* $k_1,\ldots,k_r \in \mathbb{N}_{\geq 0}$

$$(\theta_1 \circ \tilde f)^{k_1}\, s(\theta_r \circ \tilde f)^{k_r} \in [\![\tilde F]\!] \Longrightarrow \tilde f \in [\![\tilde F]\!]\,, \quad \theta_1,\ldots,\theta_r \in \Sigma\,, \quad k_1,\ldots,k_r \in \mathbb{N}_{\geq 0}\,.$$

3. Consistency of FDA with the Navier-Stockes equations

In this section we discuss the consistency issues for FDA to the system of Navier-Stokes equations in its involutive form (4) when the pressure Poisson equation (2) is incorporated.

Definition 3.1. [4, 8] *We shall say that a difference equation* $\tilde f = 0$ $(\tilde f \in \tilde{\mathcal{R}})$ *defined on grid* (7) implies the differential equation $f = 0$ $(f \in \mathcal{R})$ *and write* $\tilde f \rhd f$ *if the Taylor expansion about a grid point yields*

$$\tilde f \xrightarrow[\tau,h\to 0]{} f + O(\tau,h)$$

where $O(\tau,h)$ *denotes terms that reduce to zero when* $\tau,h \to 0$.

Definition 3.2. [4] *Given a FDA*

$$\{\, \tilde f = 0 \mid \tilde f \in \tilde F := \{\tilde f_1,\tilde f_2,\tilde f_3,\tilde f_4\,\} \subset \tilde{\mathcal{R}}\,\} \tag{9}$$

to (4)*, we shall say that FDA is* weakly consistent *or* w-consistent *with* (4)*, if*

$$(\,\forall \tilde f \in \tilde F\,)\ (\,\exists f \in F := \{\, f_1, f_2, f_3, f_4, \}\,)\ [\, \tilde f \rhd f\,]\,. \tag{10}$$

The notion of w-consistency is a natural adaptation of the notion of consistency for a single differential equation [21, 23] to differential systems (cf. [15], Sect.5.4). However, w-consistency does not guarantee suitability of a FDA for approximation of a solution to the differential system. To show this for our case of the Navier-Stokes equations (4), consider the following approximation obtained in [5]

[5]Usually one considers solutions in the universal family of difference field extensions of $\mathbb{K}$ (see [13], Sect.2.6).

$$
\left\{
\begin{aligned}
&\tilde{e}_1 := \tfrac{u^n_{j+1\,k}-u^n_{j-1\,k}}{2h} + \tfrac{v^n_{j\,k+1}-v^n_{j\,k-1}}{2h} = 0\,,\\
&\tilde{e}_2 := \tfrac{u^{n+1}_{j\,k}-u^n_{j\,k}}{\tau} + \tfrac{(u^n_{j+1\,k})^2-(u^n_{j-1\,k})^2}{2h} + \tfrac{u^n_{j\,k+1}v^n_{j\,k+1}-u^n_{j\,k-1}v^n_{j\,k-1}}{2h}\\
&\qquad + \tfrac{p^n_{j+1\,k}-p^n_{j-1\,k}}{2h} - \tfrac{1}{\mathrm{Re}}\left(\tfrac{u^n_{j+1\,k}-2u^n_{j\,k}+u^n_{j-1\,k}}{h^2} + \tfrac{u^n_{j\,k+1}-2u^n_{j\,k}+u^n_{j\,k-1}}{h^2}\right) = 0\,,\\
&\tilde{e}_3 := \tfrac{v^{n+1}_{j\,k}-v^n_{j\,k}}{\tau} + \tfrac{u^n_{j+1\,k}v^n_{j+1\,k}-u^n_{j-1\,k}v^n_{j-1\,k}}{2h} + \tfrac{(v^n_{j\,k+1})^2-(v^n_{j\,k-1})^2}{2h}\\
&\qquad + \tfrac{p^n_{j\,k+1}-p^n_{j\,k-1}}{2h} - \tfrac{1}{\mathrm{Re}}\left(\tfrac{v^n_{j+1\,k}-2v^n_{j\,k}+v^n_{j-1\,k}}{h^2} + \tfrac{v^n_{j\,k+1}-2v^n_{j\,k}+v^n_{j\,k-1}}{h^2}\right) = 0\,,\\
&\tilde{e}_4 := \tfrac{p^n_{j+1,k}-2p^n_{j,k}+p^n_{j-1,k}}{h^2} + \tfrac{p^n_{j,k+1}-2p^n_{j,k}+p^n_{j,k-1}}{h^2}\\
&\qquad + \tfrac{(u^n_{j+1,k})^2-2(u^n_{j,k})^2+(u^n_{j-1,k})^2}{h^2} + \tfrac{(v^n_{j,k+1})^2-2(v^n_{j,k})^2+(v^n_{j,k-1})^2}{h^2}\\
&\qquad + \tfrac{u^n_{j+1,k+1}v^n_{j+1,k+1}-u^n_{j+1,k-1}v^n_{j+1,k-1}-u^n_{j-1,k+1}v^n_{j-1,k+1}+u^n_{j-1,k-1}v^n_{j-1,k-1}}{2h^2}\,.
\end{aligned}
\right.
\tag{11}
$$

It is easy to see that (11) implies (4), $\tilde{e}_i \rhd f_i$ $(i = 1, \ldots, 4)$, and hence (11) is w-consistent. Consider now the $S-$polynomial (cf. Definition 2.8) associated with $\tilde{e}_1$ and $\tilde{e}_2$

$$S(\tilde{e}_1, \tilde{e}_2) := 2h\,\sigma_1 \circ \tilde{e}_2 - \tau\,\sigma_1 \circ \tilde{e}_1$$

The straightforward computation shows that this $S-$polynomial satisfies the equality

$$
\begin{aligned}
&S(\tilde{e}_1, \tilde{e}_2) - \tilde{e}_2{}^n_{j-1,k}2h + \tilde{e}_3{}^n_{j,k+1}2h - \tilde{e}_3{}^n_{j,k-1}2h + \tilde{e}_1{}^n_{j,k}\tau\\
&+ \frac{4}{\mathrm{Re}}\left(\tilde{e}_1{}^n_{j+1,k} + \tilde{e}_1{}^n_{j,k+1} - 4\tilde{e}_1{}^n_{j,k} + \tilde{e}_1{}^n_{j-1,k} + \tilde{e}_1{}^n_{j,k-1}\right) - \tilde{e}_4{}^n_{j,k}h^2 =\\
&\qquad = (u^n_{j+2,\,k})^2 - (u^n_{j+1,\,k})^2 - (u^n_{j-1,\,k})^2 + (u^n_{j-2,\,k})^2\\
&\qquad + (v^n_{j,\,k+2})^2 - (v^n_{j,\,k+1})^2 - (v^n_{j,\,k-1})^2 + (v^n_{j,\,k-2})^2\\
&+ p^n_{j+2,\,k} + p^n_{j,\,k+2} - p^n_{j+1,\,k} - p^n_{j,\,k+1} - p^n_{j-1,\,k} - p^n_{j,\,k-1} + p^n_{j-2,\,k} + p^n_{j,\,k-2}\,.
\end{aligned}
\tag{12}
$$

The left-hand side of (12) is a difference-algebraic consequence of (11) (see Definition 2.10) , and after division of the both sides in (12) by $3\,h^2$, its right-hand side implies

$$g := \Delta\, p + 2uu_{xx} + 2vv_{yy} + 2u_x^2 + 2v_y^2\,. \tag{13}$$

The differential polynomial g in (13) is not a differential-algebraic consequence of (1). This can be verified using the well known exact solution [14, 22] to the Navier-Stokes equations (1)

$$
\left\{
\begin{aligned}
&u = -e^{-\frac{2t}{\mathrm{Re}}}\cos(x)\sin(y)\,,\\
&v = e^{-\frac{2t}{\mathrm{Re}}}\sin(x)\cos(y)\,,\\
&p = -\tfrac{1}{4}e^{-\frac{4t}{\mathrm{Re}}}(\cos(2x) + \cos(2y))\,,
\end{aligned}
\right.
\tag{14}
$$

whose substitution into the right-hand side of (13) shows that g does not vanish whereas, by Definition 2.1, (14) must be a solution to any differential-algebraic consequence of (1). Therefore, FDA (11) is not suitable for numerical construction of the exact solution (14). We demonstrated this by numerical experiments in [1].

To be suitable for numerical construction of any smooth solution to the system (4), its FDA must possess the property of strong consistency formulated in the following definition.

Definition 3.3. [4] *A FDA* (9) *to the system* (4) *is* strongly consistent *or* s-consistent*, if each difference-algebraic consequence of the FDA implies a differential-algebraic consequence of* (4)*.*

$$(\forall \tilde{f} \in [\![\tilde{F}]\!]\,)\ (\,\exists f \in [\![f_1, f_2, f_3, f_4]\!]\,)\ [\,\tilde{f} \rhd f\,]\,. \tag{15}$$

Note that in the case when a PDE system under consideration possesses a conservation law in the form of differential polynomial, a good FDA to the PDE system much have a difference-algebraic consequence which imply the conservation law. An s-consistent FDA satisfies this requirement. In our case the initial PDEs admit the conservation law form (6), and any w-consistent FDA preserves this form at the discrete level.

It is clear that s-consistency implies w-consistency. The converse is generally not true, as we have shown above by example (11). In the case of linear PDE systems s-consistency admits the algorithmic verification [8] by construction of a Gröbner basis of the difference ideal generated by the polynomials occurring in FDA. Since the Navier-Stokes system (1) or (4) is nonlinear, the verification of s-consistency for its FDA is based on the following theorem proved in [4].

Theorem 3.4. [4] *A w-consistent difference approximation* (10) *to (4) is s-consistent, if and only if a standard basis $\tilde{G} \subset \tilde{\mathcal{R}}$ of the difference ideal $[\tilde{F}]$ satisfies*

$$(\forall \tilde{g} \in \tilde{G}\,)\ (\,\exists\, g \in [\![F]\!]\,)\ [\,\tilde{g} \rhd g\,]\,.$$

In contrast to linear differential systems, for nonlinear systems in general and for the Navier-Stokes equations in particular, a difference Gröbner basis may not exist, i.e. be infinite. In this situation, the algorithm described in [4, 7, 12] can be used to verify wether the intermediate $S-$polynomials, that arise in course of the algorithm imply differential-algebraic consequences of the Navier-Stokes equations. Since a $S-$polynomial is a difference consequence of the FDA under consideration, in the case of s-consistency it implies a differential polynomial that belongs to the radical differential ideal generated by the Navier-Stokes equations. This condition is necessary for the s-consistency of FDA and admits algorithmic verification.

In [4, 7, 12] the simplest forms of a Bucheberger's like algorithm were proposed for computatiing a Gröbner basis for finitely generated difference ideals.

4. Derivation of new s-consistent FDA

In this section we explain how to obtain a new s-consistent finite difference approximation to the Navier-Stokes equations (1) by means of the computation of a difference Gröbner basis. We start with a difference approximation $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ in (11) to the Navier-Stokes equations (1).

Then, we denote by $\tilde{\mathcal{I}} \subset \tilde{\mathcal{R}}$ the difference ideal which is generated by the difference polynomials corresponding to the equations (11). Aiming to obtain a time-independent equation with linear leading monomial in the variable p in order to solve numerically the FDA, we compute a difference Gröbner basis of $\tilde{\mathcal{I}}$ with respect to the lexicographic ranking with

$$\sigma_t \succ \sigma_x \succ \sigma_y \qquad p \succ u \succ v\,.$$

Precisely, we fix on the polynomial algebra $\tilde{\mathcal{R}}$ the lexicographic monomial ordering based on the following variable ordering. For all $q, r \in \{u, v, p\} \quad (p > u > v)$ we define $q^n_{j,k} \succ r^{n'}_{j',k'}$ if and only if lexicographically $(n, j, k) > (n', j', k')$ or $(n, j, k) = (n', j', k') \wedge q > r$. The result of the computation is the Gröbner basis of $\tilde{\mathcal{I}}$ given on the next page.

$$
\begin{gathered}
\tilde{g}_1 := \frac{u^n_{j+2,k+1} - u^n_{j,k+1} + v^n_{j+1,k+2} - v^n_{j+1,k}}{2h}\,,\\
\tilde{g}_2 := \frac{u^{n+1}_{j+1,k+1} - u^n_{j+1,k+1}}{\tau} - \frac{u^n_{j+2,k+1} - 4u^n_{j+1,k+1} + u^n_{j,k+1} + u^n_{j+1,k+2} + u^n_{j+1,k}}{h^2\mathrm{Re}}\\
+ \frac{(u^n_{j+2,k+1})^2 - (u^n_{j,k+1})^2 + u^n_{j+1,k+2}v^n_{j+1,k+2} - u^n_{j+1,k}v^n_{j+1,k} + p^n_{j+2,k+1} - p^n_{j,k+1}}{2h}\,,\\
\tilde{g}_3 := \frac{v^{n+1}_{j+1,k} + u^{n+1}_{j,k+1} - v^n_{j+1,k+2} - u^n_{j+2,k+1}}{\tau} - \frac{p^n_{j+1,k+1} - u^n_{j+2,k+2}v^n_{j+2,k+2}}{h}\\
\frac{4v^n_{j+1,k+2} - v^n_{j+1,k+1} + 4u^n_{j+2,k+1} - u^n_{j+1,k+1} - u^n_{j+3,k+1} - u^n_{j+2,k}}{h^2\mathrm{Re}}\\
- \frac{u^n_{j+2,k+2} + v^n_{j+2,k+2} + v^n_{j,k+2} + v^n_{j+1,k+3}}{h^2\mathrm{Re}} - \frac{u^n_{j+2,k}v^n_{j+2,k} - p^n_{j+3,k+1} - (v^n_{j+1,k+3})^2}{2h}\\
- \frac{(v^n_{j+1,k+1})^2 - p^n_{j+1,k+3} - (u^n_{j+3,k+1})^2 + (u^n_{j+1,k+1})^2 + u^n_{j,k+2}v^n_{j,k+2}}{2h}\,,\\
\tilde{g}_4 := -\frac{u^{n+1}_{j,k+2} + u^n_{j+2,k+2} - v^n_{j+1,k+1} + v^n_{j+1,k+3}}{\tau} - \frac{(v^n_{j+1,k+2})^2 - u^n_{j+2,k+1}v^n_{j+2,k+1}}{h}\\
- \frac{u^n_{j+2,k+3}v^n_{j+2,k+3}}{h} - \frac{u^n_{j+1,k+2} - 4u^n_{j+2,k+2} - v^n_{j+2,k+1} + u^n_{j+3,k+2} + v^n_{j+2,k+3}}{h^2\mathrm{Re}}\\
- \frac{u^n_{j+2,k+1} + u^n_{j+2,k+3} - v^n_{j,k+1} + v^n_{j,k+3} - v^n_{j+1,k} + v^n_{j+1,k+4} + 4v^n_{j+1,k+1} - 4v^n_{j+1,k+3}}{h^2\mathrm{Re}}\\
+ \frac{u^n_{j,k+1}v^n_{j,k+1} - u^n_{j,k+3}v^n_{j,k+3} + p^n_{j+1,k} - p^n_{j+1,k+2} + p^n_{j+1,k+4}}{2h}\\
+ \frac{p^n_{j+3,k+2} - (u^n_{j+1,k+2})^2 + (u^n_{j+3,k+2})^2 + (v^n_{j+1,k})^2 + (v^n_{j+1,k+4})^2}{2h}\,,\\
\tilde{g}_5 := \frac{p^n_{j+4,k+2} + p^n_{j+2,k+4} + p^n_{j,k+2} + p^n_{j+2,k} + (u^n_{j,k+2})^2 + (u^n_{j+4,k+2})^2}{2h}\\
+ \frac{(v^n_{j+2,k+4})^2 + (v^n_{j+2,k})^2}{2h} + \frac{v^n_{j+2,k+3} - v^n_{j+2,k+1} - u^n_{j+1,k+2} + u^n_{j+3,k+2}}{\tau}\\
- \frac{(v^n_{j+2,k+2})^2 + (u^n_{j+2,k+2})^2 - u^n_{j+1,k+1}v^n_{j+1,k+1} + u^n_{j+1,k+3}v^n_{j+1,k+3}}{h}\\
- \frac{u^n_{j+3,k+1}v^n_{j+3,k+1} - u^n_{j+3,k+3}v^n_{j+3,k+3} + 2p^n_{j+2,k+2}}{h} - \frac{4u^n_{j+1,k+2} + 4v^n_{j+2,k+1}}{h^2\mathrm{Re}}\\
+ \frac{4u^n_{j+3,k+2} + 4v^n_{j+2,k+3} + u^n_{j,k+2} + u^n_{j+1,k+1} + u^n_{j+1,k+3} - u^n_{j+3,k+1} - u^n_{j+3,k+3}}{h^2\mathrm{Re}}\\
- \frac{u^n_{j+4,k+2} - v^n_{j+1,k+1} + v^n_{j+1,k+3} - v^n_{j+2,k} + v^n_{j+2,k+4} - v^n_{j+3,k+1} + v^n_{j+3,k+3}}{h^2\mathrm{Re}}\,.
\end{gathered}
$$

Such computation is obtained by means of a Buchberger's like algorithm which is described in full detail in [7, 12]. We just mention that by applying all possible shifts to the elements $\{\tilde{g}_i\}$ one obtains a Gröbner basis of $\tilde{\mathcal{I}}$ as an ordinary ideal of $\tilde{\mathcal{R}}$. Then, this algorithm essentially reduces the s-polynomial computations of the ordinary Buchberger's algorithm up to the monoid symmetry defined by the action on $\mathcal{R}$ of the shift operators. By these methods it takes less then 3 sec to obtain $\{\tilde{g}_i\}$ with an experimental implementation in Maple. Note that this computing time has been obtained on a laptop with a four core Intel i3 at 2.20GHz and 16GB RAM. Observe now that the leading monomials of the difference Gröbner basis $\{\tilde{g}_i\}$ are all linear ones, namely

$$\begin{gathered}\operatorname{lm}(\tilde{g}_1)=u^n_{j+2,k+1}\,,\quad \operatorname{lm}(\tilde{g}_2)=u^{n+1}_{k+1,j+1}\,,\quad \operatorname{lm}(\tilde{g}_3)=v^{n+1}_{j+1,k}\,,\\ \operatorname{lm}(\tilde{g}_4)=u^{n+1}_{j,k+2}\,,\quad \operatorname{lm}(\tilde{g}_5)=p^n_{j+4,k+2}.\end{gathered}$$

Furthermore, the element g_5 is time-independent and hence it is suitable to obtain an FDA which can be easily solved numerically. Up to some minor simplifications, such FDA is given by the following equations

$$\begin{aligned}
&\frac{u^{n+1}_{j,k}-u^n_{j,k}}{\tau}+\frac{(u^n_{j+1,k})^2-(u^n_{j-1,k})^2}{2h}+\frac{v^n_{j,k+1}u^n_{j,k+1}-v^n_{j,k-1}u^n_{j,k-1}}{2h}+\frac{p^n_{j+1,k}-p^n_{j-1,k}}{2h}\\
&\qquad-\frac{1}{\mathrm{Re}}\left(\frac{u^n_{j+1,k}-2u^n_{j,k}+u^n_{j-1,k}}{h^2}+\frac{u^n_{j,k+1}-2u^n_{j,k}+u^n_{j,k-1}}{h^2}\right)=0\,,\\
&\frac{v^{n+1}_{j,k}-v^n_{j,k}}{\tau}+\frac{(v^n_{j,k+1})^2-(v^n_{j,k-1})^2}{2h}+\frac{u^n_{j+1,k}v^n_{j+1,k}-u^n_{j-1,k}v^n_{j-1,k}}{2h}+\frac{p^n_{j,k+1}-p^n_{j,k-1}}{2h}\\
&\qquad-\frac{1}{\mathrm{Re}}\left(\frac{v^n_{j+1,k}-2v^n_{j,k}+v^n_{j-1,k}}{h^2}+\frac{v^n_{j,k+1}-2v^n_{j,k}+v^n_{j,k-1}}{h^2}\right)=0\,,\\
&\qquad\frac{p^n_{j+2,k}-2p^n_{j,k}+p^n_{j-2,k}}{4h^2}+\frac{p^n_{j,k+2}-2p^n_{j,k}+p^n_{j,k-2}}{4h^2}\\
&\quad+\frac{(u^n_{j+2,k})^2-2(u^n_{j,k})^2+(u^n_{j-2,k})^2}{4h^2}+\frac{(v^n_{j,k+2})^2-2(v^n_{j,k})^2+(v^n_{j,k-2})^2}{4h^2}\\
&\quad+2\frac{u^n_{j+1,k+1}v^n_{j+1,k+1}-u^n_{j+1,k-1}v^n_{j+1,k-1}-u^n_{j-1,k+1}v^n_{j-1,k+1}+u^n_{j-1,k-1}v^n_{j-1,k-1}}{4h^2}\\
&\quad+\frac{2}{\mathrm{Re}}\frac{-u^n_{j+2,k}+4u^n_{j+1,k}-4u^n_{j-1,k}+u^n_{j-2,k}-u^n_{j+1,k+1}-u^n_{j+1,k-1}+u^n_{j-1,k+1}+u^n_{j-1,k-1}}{4h^3}\\
&+\frac{2}{\mathrm{Re}}\frac{-v^n_{j,k+2}+4v^n_{j,k+1}-4v^n_{j,k-1}+v^n_{j,k-2}-v^n_{j+1,k+1}-v^n_{j-1,k+1}+v^n_{j+1,k-1}+v^n_{j-1,k-1}}{4h^3}=0\,.
\end{aligned}$$

It is interesting to note that the last computer-generated difference equation is in fact the approximation of the following differential equation

$$\begin{gathered}(p_{xx}+p_{yy})+2(u_x^2+u_xv_y+u_yv_x+v_y^2+u(u_{xx}+v_{xy})+v(u_{xy}+v_{yy}))\\ -\frac{1}{\mathrm{Re}}(u_{xxx}+u_{xyy}+v_{xxy}+6v_{yyy})=0.\end{gathered}$$

One can check that this equation belongs to the differential ideal generated by the Navier-Stokes equations which provides the s-consistency of the above scheme.

5. Numerical implementation

Let us suppose that the square (rectangular) domain is discretized with respect to x and y in order to obtain, for each value of t, $M\times N$ gridpoints for u, v and p. At each time step the unknowns values may be obtained by means of a simple implementation of the scheme that we have introduced in the previous section.

In particular, since in the first two equations there is only one term at time $n+1$, these equations may be used to compute the unknown values of u and v explicitly.

For $j = 1, \ldots, N-1$ and $k = 1, \ldots, M-1$, from the first equation

$$\begin{aligned} u_{j,k}^{n+1} &= u_{j,k}^{n} \\ &- \frac{\tau}{2h}\left((u_{j+1,k}^{n})^2 - (u_{j-1,k}^{n})^2 + v_{j,k+1}^{n}u_{j,k+1}^{n} - v_{j,k-1}^{n}u_{j,k-1}^{n} + p_{j+1,k}^{n} - p_{j-1,k}^{n}\right) \\ &+ \frac{\tau}{h^2\mathrm{Re}}\left(u_{j+1,k}^{n} - 2u_{j,k}^{n} + u_{j-1,k}^{n} + u_{j,k+1}^{n} - 2u_{j,k}^{n} + u_{j,k-1}^{n}\right) \end{aligned}$$

while, from the second one,

$$\begin{aligned} v_{j,k}^{n+1} &= v_{j,k}^{n} \\ &- \frac{\tau}{2h}\left((v_{j,k+1}^{n})^2 - (v_{j,k-1}^{n})^2 + u_{j+1,k}^{n}v_{j+1,k}^{n} - u_{j-1,k}^{n}v_{j-1,k}^{n} + p_{j,k+1}^{n} - p_{j,k-1}^{n}\right) \\ &+ \frac{\tau}{h^2\mathrm{Re}}\left(v_{j+1,k}^{n} - 2v_{j,k}^{n} + v_{j-1,k}^{n} + v_{j,k+1}^{n} - 2v_{j,k}^{n} + v_{j,k-1}^{n}\right), \end{aligned}$$

where, for $j = 1$, $j = M-1$, $k = 1$ and $k = N-1$, the unknowns depend on the known boundary conditions. It is easy to prove that these formulae are $O(h^2)$ and $O(\tau)$ accurate.
After computing the values of u and v at time step $n+1$, the third equation proposed in the previous section may be used to compute the unknown values of p at the same time step $n+1$. For $j = 2, \ldots, N-2$ and $k = 2, \ldots, M-2$ it is necessary a rewriting in the following form

$$-p_{j-2,k}^{n+1} - p_{j,k-2}^{n+1} + 4p_{j,k}^{n+1} - p_{j,k+2}^{n+1} - p_{j+2,k}^{n+1} = b_{j,k}^{n+1},$$

where

$$\begin{aligned} b_{j,k}^{n+1} &= (u_{j+2,k}^{n+1})^2 - 2(u_{j,k}^{n+1})^2 + (u_{j-2,k}^{n+1})^2 + (v_{j,k+2}^{n+1})^2 - 2(v_{j,k}^{n+1})^2 + (v_{j,k-2}^{n+1})^2 \\ &+ 2\left(u_{j+1,k+1}^{n+1}v_{j+1,k+1}^{n+1} - u_{j+1,k-1}^{n+1}v_{j+1,k-1}^{n+1} - u_{j-1,k+1}^{n+1}v_{j-1,k+1}^{n+1} + u_{j-1,k-1}^{n+1}v_{j-1,k-1}^{n+1}\right) \\ &+ \frac{2}{h\mathrm{Re}}\Big(-u_{j+2,k}^{n+1} + 4u_{j+1,k}^{n+1} - 4u_{j-1,k}^{n+1} + u_{j-2,k}^{n+1} - u_{j+1,k+1}^{n+1} - u_{j+1,k-1}^{n+1} \\ &+ u_{j-1,k+1}^{n+1} + u_{j-1,k-1}^{n+1} - v_{j,k+2}^{n+1} + 4v_{j,k+1}^{n+1} - 4v_{j,k-1}^{n+1} + v_{j,k-2}^{n+1} \\ &- v_{j+1,k+1}^{n+1} - v_{j-1,k+1}^{n+1} + v_{j+1,k-1}^{n+1} + v_{j-1,k-1}^{n+1}\Big) \end{aligned}$$

contains known quantity. Therefore the computation of p at time step $n+1$ requires the solution of a linear system with a coefficient matrix having 5 non-zero diagonals and hence the computational cost of this system depends linearly on the number of unknowns. It is worth to note that, differently from what happens with the classical discretization of the Laplacian, in the proposed discretization the non-zero diagonals have distance 2 and $2M$ from the main one (see Fig. 1). This also means that it is necessary to combine the proposed formulae with different ones approximating the solution in the points near the boundaries.

6. Numerical tests

We have compared the new scheme (called FDA1 in the following) with one using classical discretization formulae

$$u_x(x_j, y_k, t_n) \approx \frac{u_{j+1,k}^{n} - u_{j-1,k}^{n}}{2h}, \quad u_{xx}(x_j, y_k, t_n) \approx \frac{u_{j+1,k}^{n} - 2u_{jk}^{n} + u_{j-1,k}^{n}}{h^2}$$

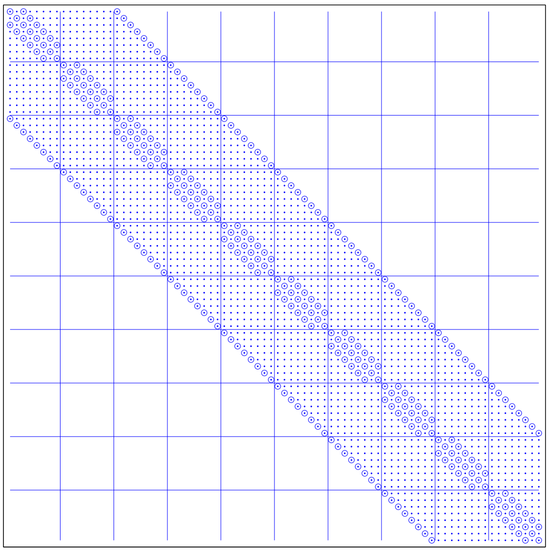

FIGURE 1. Sparsity structure of the coefficient matrix computing the unknown values of p. Circle identifies non-zero elements. Points identify non-zero elements after factorization.

of order 2 in space and

$$u_t(x_j, y_k, t_n) \approx \frac{u_{jk}^{n+1} - u_{jk}^{n}}{\tau}$$

order 1 in time (FDA2):

$$\begin{aligned}
&\frac{u_{jk}^{n+1} - u_{jk}^{n}}{\tau} + u_{jk}^{n}\frac{u_{j+1,k}^{n} - u_{j-1,k}^{n}}{2h} + v_{jk}^{n}\frac{u_{j,k+1}^{n} - u_{j,k-1}^{n}}{2h} + \frac{p_{j+1,k}^{n} - p_{j-1,k}^{n}}{2h} \\
&\qquad - \frac{1}{\mathrm{Re}}\left(\frac{u_{j+1,k}^{n} - 2u_{jk}^{n} + u_{j-1,k}^{n}}{h^2} + \frac{u_{j,k+1}^{n} - 2u_{jk}^{n} + u_{j,k-1}^{n}}{h^2}\right) = 0, \\
&\frac{v_{jk}^{n+1} - v_{jk}^{n}}{\tau} + u_{jk}^{n}\frac{v_{j+1,k}^{n} - v_{j-1,k}^{n}}{2h} + v_{jk}^{n}\frac{v_{j,k+1}^{n} - v_{j,k-1}^{n}}{2h} + \frac{p_{j,k+1}^{n} - p_{j,k-1}^{n}}{2h} \\
&\qquad - \frac{1}{\mathrm{Re}}\left(\frac{v_{j+1,k}^{n} - 2v_{jk}^{n} + v_{j-1,k}^{n}}{h^2} + \frac{v_{j,k+1}^{n} - 2v_{jk}^{n} + v_{j,k-1}^{n}}{h^2}\right) = 0, \\
&\left(\frac{u_{j+1,k}^{n} - u_{j-1,k}^{n}}{2h}\right)^2 + 2\frac{v_{j+1,k}^{n} - v_{j-1,k}^{n}}{2h}\,\frac{u_{j,k+1}^{n} - u_{j,k-1}^{n}}{2h} + \left(\frac{v_{j,k+1}^{n} - v_{j,k-1}^{n}}{2h}\right)^2 \\
&\qquad + \frac{p_{j+1,k}^{n} - 2p_{jk}^{n} + p_{j-1,k}^{n}}{h^2} + \frac{p_{j,k+1}^{n} - 2p_{jk}^{n} + p_{j,k-1}^{n}}{h^2} = 0.
\end{aligned}$$

Moreover, we have compared these schemes with the one which was proposed in [5] and studied in [1] (FDA3)

$$
\begin{aligned}
&\frac{u_{jk}^{n+1}-u_{jk}^{n}}{\tau}+\frac{{u_{j+1,k}^{n}}^2-{u_{j-1,k}^{n}}^2}{2h}+\frac{v_{j,k+1}^{n}u_{j,k+1}^{n}-v_{j,k-1}^{n}u_{j,k-1}^{n}}{2h}+\frac{p_{j+1,k}^{n}-p_{j-1,k}^{n}}{2h}\\
&\qquad-\frac{1}{\mathrm{Re}}\left(\frac{u_{j+2,k}^{n}-2u_{jk}^{n}+u_{j-2,k}^{n}}{4h^2}+\frac{u_{j,k+2}^{n}-2u_{jk}^{n}+u_{j,k-2}^{n}}{4h^2}\right)=0,\\
&\frac{v_{jk}^{n+1}-v_{jk}^{n}}{\tau}+\frac{u_{j+1,k}^{n}v_{j+1,k}^{n}-u_{j-1,k}^{n}v_{j-1,k}^{n}}{2h}\frac{{v_{j,k+1}^{n}}^2-{v_{j,k-1}^{n}}^2}{2h}+\frac{p_{j,k+1}^{n}-p_{j,k-1}^{n}}{2h}\\
&\qquad-\frac{1}{\mathrm{Re}}\left(\frac{v_{j+2,k}^{n}-2v_{jk}^{n}+v_{j-2,k}^{n}}{4h^2}+\frac{v_{j,k+2}^{n}-2v_{jk}^{n}+v_{j,k-2}^{n}}{4h^2}\right)=0,\\
&\qquad\frac{{u_{j+2,k}^{n}}^2-2{u_{j,k}^{n}}^2+{u_{j-2,k}^{n}}^2}{4h^2}+\frac{{v_{j,k+2}^{n}}^2-2{v_{j,k}^{n}}^2+{v_{j,k-2}^{n}}^2}{4h^2}\\
&+2\,\frac{u_{j+1,k+1}^{n}v_{j+1,k+1}^{n}-u_{j+1,k-1}^{n}v_{j+1,k-1}^{n}-u_{j-1,k+1}^{n}v_{j-1,k+1}^{n}+u_{j-1,k-1}^{n}v_{j-1,k-1}^{n}}{4h^2}\\
&\qquad+\frac{p_{j+2,k}^{n}-2p_{jk}^{n}+p_{j-2,k}^{n}}{4h^2}+\frac{p_{j,k+2}^{n}-2p_{jk}^{n}+p_{j,k-2}^{n}}{4h^2}=0\,.
\end{aligned}
$$

Note that the first two equations of FDA3 coincide with those proposed in this paper while the third is simpler, even if less efficient.
In the following examples we have compared these three schemes by using the following absolute/relative error formula

$$
e_g^n=\max_{j,k}\frac{|g_{j,k}^n-g(x_j,y_k,t_n)|}{1+|g(x_j,y_k,t_n)|}\,. \tag{16}
$$

where $g\in\{u,v,p\}$ and $g(x,y,t)$ belongs to the exact solution.

6.1. **Taylor decaying problem.** This is a classical Navier-Stokes problem, which is generally used to state the convergence order of the considered scheme. The exact solution (14) in $[0,2\pi]\times[0,2\pi]\times[0,6]$, where the Reynolds number $\mathrm{Re}=10^{-2}$. Fig. 2 contains the computed error for three different choices of h (error in u and v coincides). The value of $\tau=10^{-2}\ll h$ so that we are able to confirm that the order of convergence with respect to h is essentially 2 for the first two methods. On the other hand, the lower picture shows the instability of FDA3 for decreasing values of h.

6.2. **Kovasznay flow problem.** The exact solution is

$$
\begin{cases}
u:=1-e^{\lambda x}\cos(2\pi y)\,,\\
v:=\frac{\lambda}{2\pi}e^{\lambda x}\sin(2\pi y)\,,\\
p:=p_0-\frac{1}{2}e^{2\lambda x}\,.
\end{cases}
$$

in $[-1.5,1.5]\times[-2.5,2]\times[0,1]$, where $\lambda=\mathrm{Re}/2-\sqrt{\mathrm{Re}^2/4+4\pi^2}$. We have set $\mathrm{Re}=40$ and $p_0=1$.

The exact solution is independent of t but in general the numerical solution deteriorates for increasing values of the time variable. Exact solution shows oscillations in u and v with respect to the variable y (see Fig. 3 for the v-component). In Fig. 4 we depict the contour plots of the numerical solution of the v-component for $h\in\{10^{-1},5\,10^{-2},2.5\,10^{-2}\}$ with respect to the exact one (in the right-lower

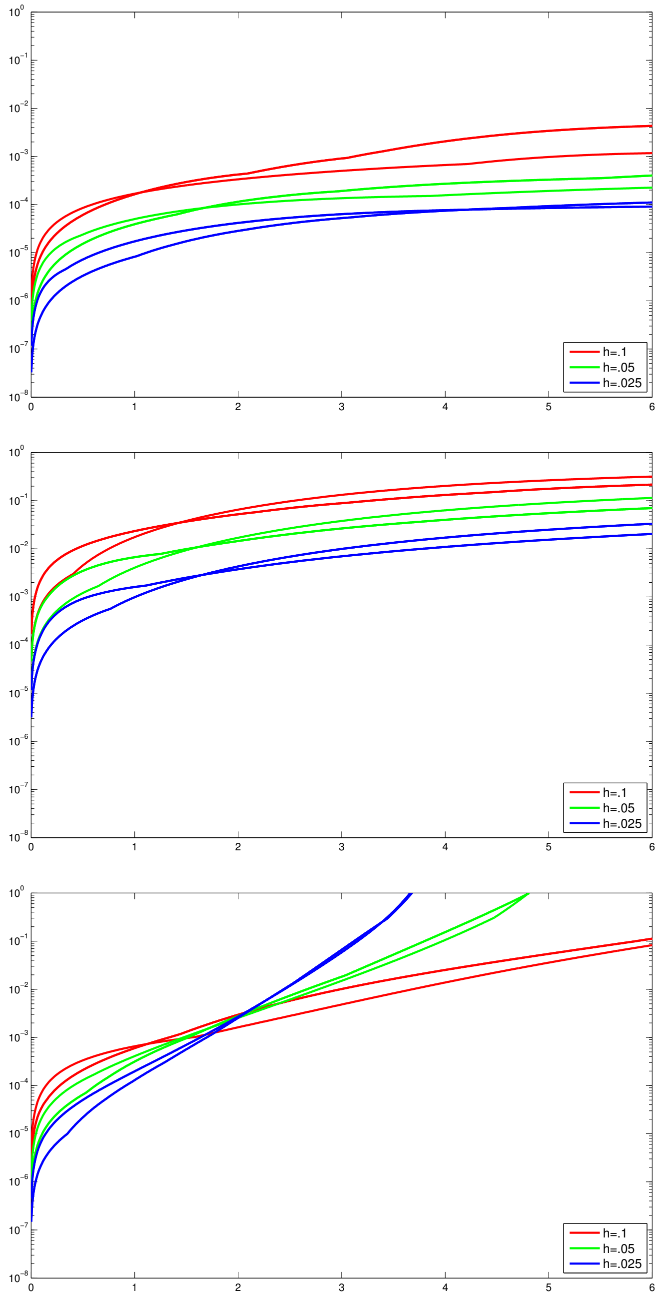

FIGURE 2. Taylor decaying problem: error with $h = .1, .05, .025$ in the computed solution with FDA1 scheme (top), second order standard discretizations FDA2 (center) and FDA3 (bottom)

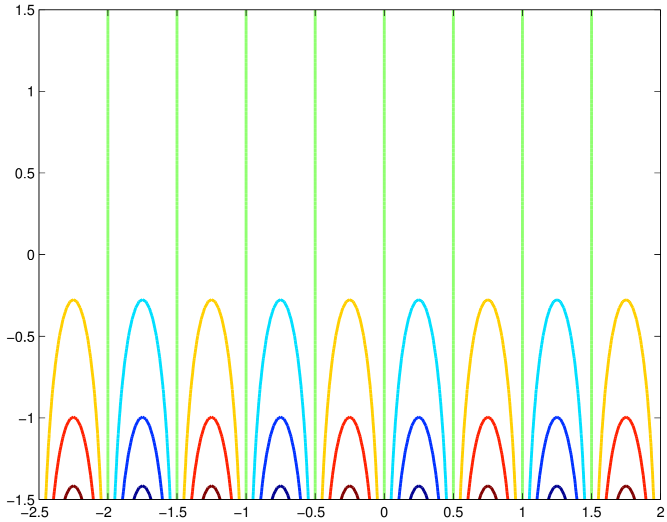

FIGURE 3. Kovasznay flow problem: exact solution for the v component

corner). We observe how strange oscillations for $h = 10^{-1}$ disappear when smaller stepsize are considered

In Table 1 we show the error in the three components of the solution.

h	Error in u	Error in v	Error in p
1e-1	5.74e-01	4.26e-01	4.16e-01
5e-2	2.36e-01	1.44e-01	1.17e-01
2.5e-2	6.94e-02	3.45e-02	6.26e-02

TABLE 1. Kovasznay flow problem: computed error with the FDA1 scheme for different values of h.

7. Conclusions

In this paper we introduce a new s-consistent finite difference approximation to the Navier-Stokes equations. By using the symbolic methods in [7, 12] our construction is obtained by computing the difference Gröbner basis for the ideal generated by the three difference polynomials which were derived earlier in [5], as a part of the s-consistent approximation to the involutive form of the Navier-Stokes equations. Those three difference polynomials are w-consistent with the continuity equation and the proper Navier-Stokes equations.

The obtained Gröbner basis contains two more difference polynomials where one of them is equivalent to the pressure Poisson equation modulo the difference ideal which is generated by the proposed basis. We have added the equation corresponding to this element of the Gröbner basis to the initial three difference equations

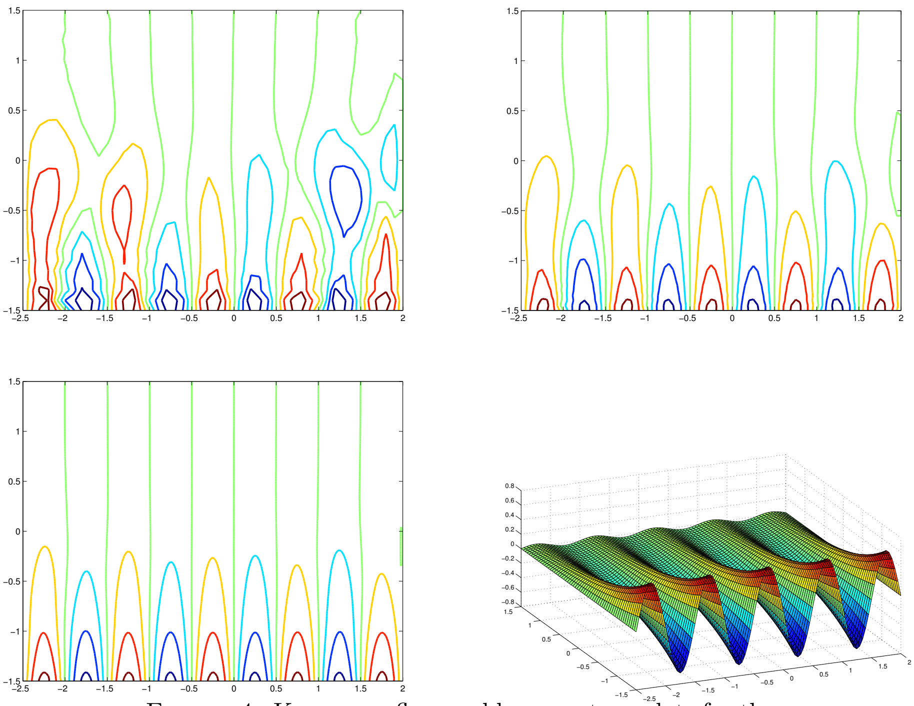

FIGURE 4. Kovasznay flow problem: contour plots for the numerical solutions with $h = .1, .05, .025$ and the exact solution at $t = 1$ for the v component of the solution

and we have performed some mutual simplification in the obtained finite difference approximation. We have finally shown that the new difference system, as approximation to the Navier-Stokes equations, is s-consistent and has a very good numerical behavior.

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