This is the Pre-print of the paper
A. Loiudice, Critical growth problems with singular nonlinearities on Carnot groups
Nonlinear Anal. 126 (2015), 415-436
DOI: https://doi.org/10.1016/j.na.2015.06.010

Critical growth problems with singular nonlinearities on Carnot groups

Annunziata Loiudice
Dipartimento di Matematica
Università degli Studi di Bari
Via Orabona, 4 - 70125 Bari (Italy)

Abstract

We provide regularity, existence and non existence results for the semilinear subelliptic problem with critical growth $-\Delta_{\mathbb{G}} u = \psi^\alpha \frac{|u|^{2^*(\alpha)-2}u}{d(\xi)^\alpha} + \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, where $\Delta_{\mathbb{G}}$ is a sublaplacian on a Carnot group $\mathbb{G}$, $0 < \alpha < 2$, $2^*(\alpha) = 2(Q-\alpha)/(Q-2)$, Ω is a bounded domain of $\mathbb{G}$, d is the natural gauge associated with the fundamental solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$ and $\psi := |\nabla_{\mathbb{G}} d|$, $\nabla_{\mathbb{G}}$ being the subelliptic gradient associated to $\Delta_{\mathbb{G}}$, λ is a real parameter.

2000 *Mathematics Subject Classification*: 35H20, 35J70.
Keywords: subelliptic critical problem, singular nonlinearity, Hardy-Sobolev extremal, Carnot group.

1 Introduction

In this paper we study the critical semilinear boundary value problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \psi^\alpha \dfrac{|u|^{2^*(\alpha)-2}u}{d(\xi)^\alpha} + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Delta_{\mathbb{G}}$ is the Sublaplacian operator on a Carnot group $\mathbb{G}$, d is the natural gauge associated with the fundamental solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$, $0 < \alpha < 2$ and $2^*(\alpha) = 2(Q-\alpha)/(Q-2)$ is the corresponding critical exponent, Q being the homogeneous dimension of the space $\mathbb{G}$. Moreover, ψ is the weight function defined as $\psi := |\nabla_{\mathbb{G}} d|$, where $\nabla_{\mathbb{G}}$ is the subelliptic gradient associated to $\Delta_{\mathbb{G}}$. Here, Ω is a bounded domain of $\mathbb{G}$ and λ is a real parameter.

We look for weak solutions of (1.1) in the Folland-Stein space $S^1_0(\Omega)$, defined as the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{S^1_0(\Omega)} := \left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \right)^{1/2}.$$

When $\Omega = \mathbb{G}$, we shall simply denote $S^1(\mathbb{G}) = S^1_0(\mathbb{G})$.

In the past years, a great deal of interest has been paid to semilinear problems with critical nonlinearities arising in the context of Stratified groups (see e.g. [20], [32], [33], [7], [21], [3], [36]), the greatest part of this literature concerning autonomous nonlinearities.

On the other hand, in the Euclidean elliptic setting, singular nonlinear problems with critical growth of the type (1.1), have been studied by many authors (see e.g. [26], [25], [5], [16], [6], [27], just to cite the ones that are somehow related to the present one).

In this paper, we begin the study of Brezis-Nirenberg type singular problems in the context of Carnot groups, starting from the case of interior singularity (i.e. $0 \in \Omega$).

The variational formulation of pb. (1.1) stands on the validity of the following subelliptic *Hardy-Sobolev type inequality*, which holds in any Carnot group of homogeneous dimension $Q \geq 3$. Assume that $0 \leq \alpha < 2$; then, there exists a positive constant $C = C(\alpha, Q)$ such that

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi \geq C \left(\int_{\mathbb{G}} \psi^\alpha \frac{|u|^{2^*(\alpha)}}{d(\xi)^\alpha} \, \mathrm{d}\xi \right)^{\frac{Q-2}{Q-\alpha}}, \quad \forall u \in C_0^\infty(\mathbb{G}) \tag{1.2}$$

where $2^*(\alpha) = \frac{2(Q-\alpha)}{Q-2}$ and $\psi = |\nabla_{\mathbb{G}} d|$.

We emphasize that the weight function ψ appearing in the l.h.s. of (1.2) is constantly equal to 1 in the Euclidean canonical case. The above inequality can be easily obtained by combining the appropriate Sobolev and Hardy-type inequalities on Carnot groups (see Section 2).

The main difficulty in facing nonlinear critical problems of type (1.1) is the lack of compactness in the related Hardy-Sobolev embedding, due to the invariance of the norms in (1.2) with respect to the following non compact group of rescalings

$$u_\lambda(\xi) = \lambda^{\frac{Q-2}{2}} u(\delta_\lambda(\xi)), \quad \lambda > 0, \tag{1.3}$$

where δ_λ denotes the natural dilations of the group.

In this paper, we investigate existence and non existence of nonnegative weak solutions to pb. (1.1) on bounded domains of $\mathbb{G}$. To get non existence results, a deep regularity analysis at 0 is performed.

First of all, we consider the case $\lambda = 0$. By means of suitable integral identities of Pohozaev-type, we prove that, if $\mathbb{G}$ is a Carnot group of step two, then the critical problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \psi^\alpha \dfrac{|u|^{2^*(\alpha)-2} u}{d(\xi)^\alpha} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

does not admit nonnegative nontrivial solutions, sufficiently regular up to the boundary, when Ω is a bounded starshaped domain about the origin with respect to the dilations of the group.

We recall that Pohozaev-type identities in the Stratified subelliptic context were first introduced by Lanconelli and Garofalo in [20] in the Heisenberg group $\mathbb{H}^n$ and extended by Garofalo and Vassilev [21] to general Carnot groups. In the just cited papers, many non-existence results were proved for Yamabe-type problems (i.e. for the case $\alpha = 0$ in the above equation), both on bounded and unbounded domains (see also [32], [33], [3]).

In the Euclidean context, Pohozaev identities for singular problems of the type (1.4) can be found e.g. in [26], [16], [5], [27].

In the present Carnot case, some additional difficulties arise in implementing Pohozaev type results, in connection with the lack of regularity of solutions at 0.

As stated in Proposition 3.1, a weak solution $u \in S_0^1(\Omega)$ to problem (1.4) satisfies

$$u \in \Gamma^{2,\gamma}_{loc}(\Omega \setminus \{0\}) \cap \Gamma^{\beta}_{loc}(\Omega),$$

for some $\gamma, \beta \in (0,1)$, where the above spaces are the appropriate Folland-Stein spaces (see Section 2 for definitions). For general $0 < \alpha < 2$, one cannot expect more than Hölder regularity at the origin, as already pointed out in the Euclidean case.

Now, observe that, in the Carnot case, the infinitesimal generator Z of the dilations $\{\delta_\lambda\}_{\lambda>0}$, by means of which the notion of starshapedness is defined (see Definition 4.2), involves commutators up to maximum length of the vector fields X_j, such that $\nabla_{\mathbb{G}} = (X_1, \ldots, X_m)$. Henceforth, summability properties of Zu at 0, being $u \in S_0^1(\Omega)$ a weak solution to our pb. (1.4), cannot be directly deduced by the condition $|\nabla_{\mathbb{G}} u| \in L^2$. So, we are lead to study the behavior at 0 of the higher-layer derivatives of u.

This is the main technical difficulty with respect to the Euclidean case. To obtain the needed estimates, we use a pointwise approach, inspired by the methods introduced by Lanconelli and Uguzzoni in [32], [33] in the non-singular case $\alpha = 0$.

Our analysis is confined to Carnot groups of step two. In this context, denoted by $\mathfrak{g} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$ the Lie algebra of $\mathbb{G}$, where $\mathfrak{G}_1 = span\{X_1, \ldots, X_m\}$ and $\mathfrak{G}_2 = [\mathfrak{G}_1, \mathfrak{G}_1]$, by means of a careful asymptotic analysis, we get that, if u is a weak solution of problem (1.4), and $Y \in \mathfrak{G}_2$, then

$$|Yu(\xi)| = \mathcal{O}(d(\xi)^{-2}), \quad \text{as} \quad d(\xi) \to 0. \tag{1.5}$$

A key ingredient in our analysis is to estimate the local behavior at 0 of Yw, where $w = \Gamma * f$, Γ being the fundamental solution of $\Delta_{\mathbb{G}}$ and $f = f(\cdot, u)$ is the r.h.s. of equation (1.4), where u is trivially extended outside of Ω. We remark that, for the function w, the sharp estimate $|Yw(\xi)| = \mathcal{O}(d(\xi)^{-\alpha})$, as $d(\xi) \to 0$, is obtained.

Thus, by means of the technical information (1.5), we can implement Pohozaev-type identities for our singular problem, in order to get non-existence of solutions.

Our main results are Theorem 4.5 and Theorem 4.6. In the first theorem, we prove non existence of non trivial non-negative solutions for pb. (1.4), Γ^2-regular up to the boundary of Ω, when Ω is bounded and δ_λ-starshaped about the origin. In the second one, under the hypothesis that Schauder type boundary estimates hold at non-characteristic points (as in the case $\mathbb{G} = \mathbb{H}^n$), reasoning as in [21] we get non-existence results under the weaker assumptions of boundedness of $\nabla_{\mathbb{G}} u$ and Zu up to the boundary.

Next, we investigate the existence of solutions for problem (1.1) when $\lambda > 0$. The main difficulty one encounters when dealing with the subelliptic singular problem (1.1) is that the explicit form of the Hardy-Sobolev extremals in the Carnot setting is not known, even for the Heisenberg group $\mathbb{H}^n$. This lack of information seems to make the known techniques, namely the Brezis-Nirenberg methods in [4], not applicable to our context. Nevertheless, as already recognized by the author in [36], where the case $\alpha = 0$ was studied, this difficulty can be overcome, since the real ingredient which is needed to perform asymptotic espansions of Brezis-Nirenberg type is the knowledge of the asymptotic behaviour of Hardy-Sobolev minimizers at ∞.

Now, in the case $0 < \alpha < 2$, by a suitable adaptation of Lions' concentration-compactness argument [35], the existence of Hardy-Sobolev extremals can be proved (see Section 3). Moreover, by the general asymptotic results in [37], we deduce that such extremals behave at ∞ exactly as the fundamental solution Γ of $\Delta_{\mathbb{G}}$.

Thus, by using this facts, we can prove the following theorem, where λ_1 denotes the first eigenvalue of $-\Delta_{\mathbb{G}}$ in Ω with Dirichlet boundary conditions.

Theorem 1.1. *Let $\mathbb{G}$ be a Carnot group of homogeneous dimension $Q > 3$ and let $\Omega \subset \mathbb{G}$ be a bounded domain, $0 \in \Omega$. Then, problem (1.1) admits at least one positive solution $u \in S_0^1(\Omega)$ for any $0 < \lambda < \lambda_1$.*

We explicitly remark that the existence result is not affected by the power α of the singularity, since it only depends on the involved operator and the criticality of the pair $(\alpha, 2^*(\alpha))$. Moreover, considering that, for $Q \leq 3$, then $\mathbb{G}$ is necessarily the ordinary Euclidean space $(\mathbb{R}^N, +)$, we can observe that, except for the trivial case $(\mathbb{R}^3, +)$, no *critical dimensions* in the sense of Pucci and Serrin appear for this problem, i.e. there are no homogeneous space dimensions for which problem (1.1) does not admit any non trivial solution in a right neighborhood of $\lambda = 0$.

The paper is organized as follows. In Section 2, we recall the main features of the functional setting of Carnot group and the main notations. Section 3 is devoted to regularity results; in particular, we provide the estimates of the asymptotic behavior at 0 of the second-layer derivatives of solutions, needed to implement Pohozaev identity, in the context of step two Carnot groups. In Section 4 we prove non-existence results for our singular problem, by means of Pohozaev-type arguments. In Section 5 we investigate qualitative properties of Hardy-Sobolev extremals, which are needed in the existence result. Finally, in Section 6 we prove Theorem 1.1.

2 The functional setting

We briefly recall the relevant definitions and notations related to the Carnot groups functional setting. For a complete treatment, we refer to the wide monograph [2] and the classical papers [17], [18].

A *Carnot group* (or *Stratified group*) $(\mathbb{G}, \circ)$ is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, namely a decomposition $\mathfrak{g} = \bigoplus_{j=1}^r \mathfrak{G}_j$, such that $[\mathfrak{G}_1, \mathfrak{G}_j] = \mathfrak{G}_{j+1}$ for $1 \leq j < r$, and $[\mathfrak{G}_1, \mathfrak{G}_r] = \{0\}$. The number r is called the *step* of the group $\mathbb{G}$. The integer $Q = \sum_{i=1}^r i \dim(\mathfrak{G}_i)$ is called the *homogeneous dimension* of $\mathbb{G}$. We shall assume throughout that $Q \geq 3$. Note that, if $Q \leq 3$, then $\mathbb{G}$ is necessarily the ordinary Euclidean space $\mathbb{G} = (\mathbb{R}^N, +)$.

By means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map (which we shall assume throughout), it is not restrictive to suppose that $\mathbb{G}$ be a homogeneous Lie group on $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \ldots \times \mathbb{R}^{N_r}$, with $N_i = \dim(\mathfrak{G}_i)$, equipped with a family of group-automorphisms (called *dilations*) δ_λ of the form

$$\delta_\lambda(\xi) = (\lambda\, \xi^{(1)}, \lambda^2\, \xi^{(2)}, \cdots, \lambda^r\, \xi^{(r)}), \tag{2.1}$$

where $\xi^{(j)} \in \mathbb{R}^{N_j}$ for $j = 1, \ldots, r$. Let $m := N_1$ and let $X_1, \ldots, X_m$ be the set of left invariant vector fields of $\mathfrak{G}_1$ that coincide at the origin with the first m partial derivatives. The second order differential operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{m} X_i^2$$

is called the canonical *sub-Laplacian* on $\mathbb{G}$. We shall denote by

$$\nabla_{\mathbb{G}} = (X_1, \ldots, X_m) \tag{2.2}$$

the related subelliptic gradient. Note that $\Delta_{\mathbb{G}}$ is left-translation invariant w.r.t. the group action and δ_λ-homogeneous of degree two. In other words, $\Delta_{\mathbb{G}}(u \circ \tau_\xi) = \Delta_{\mathbb{G}} u \circ \tau_\xi$, $\Delta_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda^2 \Delta_{\mathbb{G}} u \circ \delta_\lambda$. Moreover, due to the stratification condition, the Lie algebra generated by $X_1, \ldots, X_m$ is the whole $\mathfrak{g}$, and therefore it is everywhere of rank N; therefore, the sublaplacian operator $\Delta_{\mathbb{G}}$ satisfies the well-known Hörmander's hypoellipticity condition.

We recall that, if $\Omega \subset \mathbb{G}$ is a smooth open set, the *characteristic set* of Ω with respect to the system of vector fields (2.2) is defined as

$$\Sigma := \{\xi \in \partial\Omega \,|\, X_i(\xi) \in T_\xi(\partial\Omega), i = 1, \ldots, m\}, \tag{2.3}$$

where $T_\xi(\partial\Omega)$ denotes the tangent space to $\partial\Omega$ at the point ξ. For many examples and properties of the characteristic set, see e.g. [13, Chapter 3]. We shall deal with the characteristic set of the domain Ω in Section 3 in connection with the problem of regularity of solutions up to the boundary.

When $Q \geq 3$, Carnot groups possess the following property: there exists a suitable homogeneous norm d on $\mathbb{G}$ such that

$$\Gamma(\xi) = \frac{C}{d(\xi)^{Q-2}} \tag{2.4}$$

is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$ (see [17]). By definition, a *homogeneous norm* on $\mathbb{G}$ is a continuous function $d : \mathbb{G} \to [0, +\infty)$, smooth away from the origin, such that $d(\delta_\lambda(\xi)) = \lambda\, d(\xi)$, for every $\lambda > 0$ and $\xi \in \mathbb{G}$, $d(\xi^{-1}) = d(\xi)$ and $d(\xi) = 0$ iff $\xi = 0$. Moreover, if we define $d(\xi, \eta) := d(\eta^{-1} \circ \xi)$, then d is a pseudo-distance on $\mathbb{G}$. In particular, d satisfies the pseudo-triangular inequality

$$d(\xi, \eta) \leq \beta(d(\xi, \zeta) + d(\zeta, \eta)), \quad \xi, \eta, \zeta \in \mathbb{G}, \tag{2.5}$$

for a suitable constant β. Throughout the paper, we shall denote by d the homogeneous norm associated to the fundamental solution of the sub-Laplacian by (2.4). We shall indicate by $B_r(\xi) = B_d(\xi, r)$ the d-ball with center at ξ and radius r.

A fundamental rôle in the functional analysis on Carnot groups is played by the following Sobolev-type inequality due to Folland and Stein [17]: there exists a positive constant $S = S(\mathbb{G})$ such that

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \geq S \left(\int_{\mathbb{G}} |u|^{2^*} \,\mathrm{d}\xi \right)^{2/2^*} \qquad \forall\, u \in C_0^\infty(\mathbb{G}), \tag{2.6}$$

where $2^* = 2Q/(Q-2)$ is the critical exponent in this context.

Moreover, on any Carnot group $\mathbb{G}$ the following Hardy-type inequality holds:

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \geq \left(\frac{Q-2}{2} \right)^2 \int_{\mathbb{G}} \psi^2 \frac{|u|^2}{d(\xi)^2} \,\mathrm{d}\xi, \quad \forall u \in C_0^\infty(\mathbb{G}), \tag{2.7}$$

where d is the gauge associated with the fundamental solution of $\Delta_{\mathbb{G}}$ on $\mathbb{G}$ and $\psi = |\nabla_{\mathbb{G}} d|$. The preceding inequality was firstly proved by Lanconelli and Garofalo in [19] for the Heisenberg group (see also [9]). Then, it has been extended to all Carnot groups (see e.g. [10]). We recall that the constant in the r.h.s. of the formula (2.7) is sharp and it is never attained (see e.g. [10]). We explicitly note that the weight function ψ appearing in the r.h.s. of (2.7) is constant if and only if $\mathbb{G}$ is the Euclidean group (see [2, Prop. 9.8.9]). Moreover, ψ is δ_λ-homogeneous of degree 0 and ψ^2 is a smooth function out of the origin.

Now, if one is not interested in the best constant, by combining the above Hardy-type inequality (2.7) with the Folland-Stein Sobolev inequality (2.6), one immediately gets the Hardy-Sobolev inequality (1.2). For a deep treatment of Hardy-Sobolev inequalities in the general context of Carnot-Caratheodory spaces, we refer to [14].

In what follows, we shall use the following spaces. For an open set $\Omega \subset \mathbb{G}$, we shall denote by $\Gamma^2(\Omega)$ the Folland-Stein space of all continuous functions $u \in C(\Omega)$ such that $X_j u$, $X_i X_j u \in C(\Omega)$, for $i, j = 1, \dots, m$. Analogously we shall denote by $\Gamma^{k,\beta}(\overline{\Omega})$, $0 < \beta < 1$, $k \in \mathbb{N} \cup \{0\}$, the Folland-Stein Hölder spaces (see Folland [17]), and by $\Gamma^{k,\beta}_{loc}(\Omega)$ the space of functions which belongs to $\Gamma^{k,\beta}(D)$, for any compact subset D of Ω.

Moreover, $L^{p,\infty}$, $p \geq 1$, will denote the classical weak-L^p space and $L^{p',1}$ its conjugate space in the framework of Lorentz spaces (see e.g. [1], [23] for definitions and properties).

3 Some regularity results

In this section we provide some regularity properties of weak solutions to problem (1.4). In particular, we estimate the asymptotic behaviour at 0 of the second-layer derivatives of solutions, needed to implement Pohozaev identities. We shall use a suitable adaptation of the techniques introduced by Lanconelli and Uguzzoni in [32], [33].

Throughout this section, Ω will be an arbitrary open subset of $\mathbb{G}$. We begin by stating the Hölder regularity properties of weak solutions. The starting point is the analysis of the L_p-properties of solutions performed by the author in [37].

Proposition 3.1. *Let $\mathbb{G}$ be a Carnot group and let Ω be an arbitrary open set of $\mathbb{G}$, $0 \in \Omega$. If $u \in S^1_0(\Omega)$ is a weak solution of problem*

$$\begin{cases} -\Delta_{\mathbb{G}} u &= \psi^\alpha \dfrac{|u|^{2^*(\alpha)-2} u}{d(\xi)^\alpha} \quad \text{in } \Omega \\ \qquad u &= 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{cases} \tag{3.1}$$

then

$$u \in \Gamma^{2,\gamma}_{loc}(\Omega \setminus \{0\}) \cap \Gamma^{\beta}_{loc}(\Omega),$$

for some $\gamma, \beta \in (0,1)$. Moreover, if Ω satisfies the following geometric condition

$$\exists\, \delta, r_0 > 0 : |B_d(\xi, r) \setminus \Omega| \geq \delta |B_d(\xi, r)| \quad \forall\, \xi \in \partial\Omega, \forall r \in (0, r_0), \tag{3.2}$$

then, u is Hölder continuos up to the boundary of Ω.

Proof. By Proposition 4.4 in [37], we know that $u \in L^p$ for any $p \in (\frac{2^*}{2}, \infty]$. Then, if we consider u as a solution of the equation $-\Delta_{\mathbb{G}} u = Vu$, with $V = \psi^\alpha \frac{|u|^{2^*(\alpha)-2}}{d^\alpha}$, from $u \in L^p$ for

$p > 2^*$, and taking into account that $\psi \in L^\infty$, we get that

$$V \in L^p, \text{ for some } p > \frac{Q}{2}. \tag{3.3}$$

The summabity condition (3.3) on V ensures, by the classical results by Stampacchia [40], suitably extended to the subelliptic context (see e.g. [8]), that u is locally Hölder continuous in Ω, i.e.

$$u \in \Gamma^{\beta}_{loc}(\Omega), \text{ for some } \beta \in (0,1). \tag{3.4}$$

Moreover, denoted by f the r.h.s. of the equation in (3.1), by (3.4) and being ψ^α, $d^{-\alpha}$ locally Hölder continuous out of the origin, it follows that $f \in \Gamma^{\gamma}_{loc}(\Omega \setminus \{0\})$, for some $\gamma \in (0,1)$. Henceforth, by Folland [17, Theor. 6.1], $u \in \Gamma^{2,\gamma}_{loc}(\Omega \setminus \{0\})$.

Finally, if Ω satisfies (3.2), the Hölder continuity up to the boundary can be proved by applying an analogue of Moser's iteration technique (see e.g. [22], Chapter 8) suitably adapted to the subelliptic case. We omit the details. $\square$

In order to estimate the second layer derivatives of u at 0, we shall confine our analysis to Carnot groups of step two. We recall that, in this case, the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ admits the stratification $\mathfrak{g} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$, where $\mathfrak{G}_1 = span\{X_1, \ldots, X_m\}$ and $\mathfrak{G}_2 = [\mathfrak{G}_1, \mathfrak{G}_1]$. Our result is the following.

Theorem 3.2. *Let $\mathbb{G}$ be a Carnot group of step two. Suppose that $\Omega \subset \mathbb{G}$ is an open set, $0 \in \Omega$, and let $u \in S^1_0(\Omega)$ be a weak solution of pb.* (3.1)*. Then, for every $Y \in \mathfrak{G}_2$, we have*

$$|Yu(\xi)| = \mathcal{O}(d(\xi)^{-2}) \quad \text{as } d(\xi) \to 0.$$

In order to prove the above theorem, we shall need some preliminary results. Let us begin by introducing some notations. If $u \in S^1_0(\Omega)$ is a solution of pb. (3.1), and we set u to be 0 outside Ω, denoted by

$$f = \psi^\alpha \frac{|u|^{2^*(\alpha)-2} u}{d^\alpha}, \tag{3.5}$$

we define the function

$$w = \Gamma * f : \mathbb{G} \to \mathbb{R}, \quad w(\xi) = \int_{\mathbb{G}} \Gamma(\xi, \xi') f(\xi') \, \mathrm{d}\xi', \tag{3.6}$$

where Γ denotes the fundamental solution of $-\Delta_{\mathbb{G}}$ introduced in (2.4).

Note that, by $\psi \in L^\infty$, $d^{-\alpha} \in L^{Q/\alpha,\infty}$, and being $u \in L^\infty$ and $u = \mathcal{O}(d^{-(Q-2)})$ at ∞ (see [37]), (and so $f = \mathcal{O}(d^{-(Q+2-\alpha)})$ at ∞), it follows that

$$f \in L^1(\mathbb{G}) \cap L^{Q/\alpha,\infty}(\mathbb{G}). \tag{3.7}$$

Therefore, since $\Gamma \in L^{2^*/2,\infty}(\mathbb{G})$, by Young's inequality for weak-L^p spaces (see e.g. [23]) we deduce that

$$w \in L^{2^*/2,\infty}(\mathbb{G}) \cap L^\infty(\mathbb{G}).$$

Indeed, from $f \in L^1(\mathbb{G})$ and $\Gamma \in L^{2^*/2,\infty}(\mathbb{G})$, one gets $\Gamma * f \in L^{2^*/2,\infty}(\mathbb{G})$; moreover, $f \in L^{Q/\alpha,\infty}(\mathbb{G}) \cap L^1(\mathbb{G})$ implies, by interpolation, that $f \in L^{Q/2,1}(\mathbb{G})$, hence $\Gamma * f \in L^\infty$ (see [1], Remark 7.29). Moreover, w weakly solves

$$-\Delta_{\mathbb{G}} w = f \quad \text{in } \mathbb{G}.$$

Concerning the regularity of w, taking into account that u is locally Hölder continuous in Ω, it follows that

$$f \in \Gamma^{\gamma}_{loc}(\Omega \setminus \{0\}), \tag{3.8}$$

for some $\gamma \in (0,1)$. Then, from Folland [17, Theor. 6.1], we deduce that $w \in \Gamma^{2,\gamma}_{loc}(\Omega \setminus \{0\})$.

Moreover, if Ω satisfies the geometric condition (3.2), then the trivial extension of u is Hölder continuous in the whole $\mathbb{G}$. Henceforth, in this case, $f \in \Gamma^{\gamma}_{loc}(\mathbb{G} \setminus \{0\})$ and so $w \in \Gamma^{2,\gamma}_{loc}(\mathbb{G} \setminus \{0\})$.

Now, in order to estimate the behavior of the second-layer derivatives of u at 0, we shall first estimate such derivatives for the convolution function w, then we shall consider their behaviour for the function

$$v = w - u, \tag{3.9}$$

which satisfies

$$\Delta_{\mathbb{G}} v = 0 \ \text{ in } \Omega \setminus \{0\}, \qquad v = w \ \text{ on } \partial\Omega.$$

To begin with, we prove the following general result, which we shall apply to represent the derivatives of w in the set $U = \Omega \setminus \{0\}$.

Lemma 3.3. *Let $\mathbb{G}$ be a Carnot group of step two and U be an arbitrary open set of $\mathbb{G}$. Let $f \in L^p(\mathbb{G})$, for some $p \in [1, \frac{Q}{2}[$. Suppose moreover that $f|_U \in \Gamma^{\gamma}_{loc}(U)$ for some $\gamma \in (0,1)$. If we set $w = \Gamma * f$, then $w|_U \in \Gamma^{2,\gamma}_{loc}(U)$ and for every $Y \in \mathfrak{G}_2$*

$$Yw(\xi_0) = \int_{B_d(\xi_0,r)} Y\Gamma(\xi_0,\xi')(f(\xi') - f(\xi_0))\,\mathrm{d}\xi' + \int_{\mathbb{G}\setminus B_d(\xi_0,r)} Y\Gamma(\xi_0,\xi') f(\xi')\,\mathrm{d}\xi', \tag{3.10}$$

for any $\xi_0 \in U$ and $r > 0$ such that $B_d(\xi_0, r) \Subset U$.

Proof. We shall follow the line of the proof given in [32], [33]. We stress that, in the just cited papers, being f Hölder continuous in the whole $\mathbb{G}$, the above representation formula holds for any $\xi_0 \in \mathbb{G}$ and without any restriction on the radius r.

The regularity of w follows by Folland [17, Theorem 5.13]. Let $\xi_0 \in U$ and $r > 0$ such that $B_d(\xi_0, r) \Subset U$. We define

$$B_0 = B_d(\xi_0, r), \quad B = B_d\left(\xi_0, \frac{r}{2\beta}\right)$$

$$f_0 = f\chi_{B_0}, \quad f_1 = f - f_0,$$

where β is the constant appearing in the pseudo-triangular inequality (2.5) and χ_{B_0} is the characteristic function of the set B_0. Let $\eta \in C^\infty(\mathbb{R}, [0,1])$ be such that $\eta \equiv 0$ in $[0,1]$, $\eta \equiv 1$ in $[2, +\infty[$, and for every $\varepsilon > 0$ let us set $\eta_\varepsilon = \eta(d/\varepsilon) \in C^\infty(\mathbb{G})$; moreover, let $\eta_\varepsilon(\xi, \xi') = \eta_\varepsilon(\xi'^{-1} \circ \xi)$. We set

$$w_0 = \Gamma * f_0, \quad w_1 = \Gamma * f_1, \quad w_{0,\varepsilon} = (\Gamma\eta_\varepsilon) * f_0.$$

Note that, since $f \in L^\infty(B)$, it is immediate to verify that

$$\lim_{\varepsilon \to 0} w_{0,\varepsilon} = w_0 \qquad \text{uniformly on the compact sets of } B. \tag{3.11}$$

Moreover, $w_{0,\varepsilon} \in C^\infty(B)$. Taking into account that, in a group of step two, for any regular function g, we have

$$Y_\xi(g(\xi'^{-1} \circ \xi) = -Y_{\xi'}(g(\xi'^{-1} \circ \xi)),$$

for every $\xi \in B$ and $\varepsilon < r/(4\beta)$, we get

$$\begin{aligned}
Yw_{0,\varepsilon}(\xi) &= \int_{B_0} Y_\xi \left(\Gamma\eta_\varepsilon(\xi,\xi')\right) f_0(\xi')\,\mathrm{d}\xi' \\
&= \int_{B_0} Y_\xi \left(\Gamma\eta_\varepsilon(\xi,\xi')\right)(f_0(\xi') - f_0(\xi))\,\mathrm{d}\xi' - f_0(\xi)\int_{B_0} Y_{\xi'}\left(\Gamma\eta_\varepsilon(\xi,\xi')\right)\mathrm{d}\xi' \\
&= \int_{B_0} Y\left(\Gamma\eta_\varepsilon(\xi,\xi')\right)(f_0(\xi') - f_0(\xi))\,\mathrm{d}\xi' - f_0(\xi)\int_{\partial B_0} \Gamma(\xi,\xi') < Y(\xi'), \nu(\xi') > \,\mathrm{d}\sigma(\xi').
\end{aligned}$$

Let us define $\overline{w}_0 : B \to \mathbb{R}$ as

$$\overline{w}_0(\xi) := \int_{B_0} Y\Gamma(\xi,\xi')(f_0(\xi') - f_0(\xi))\,\mathrm{d}\xi' - f_0(\xi)\int_{\partial B_0} \Gamma(\xi,\xi') < Y(\xi'), \nu(\xi') > \,\mathrm{d}\sigma(\xi').$$

We claim that $\overline{w}_0$ is well-posed on B and that

$$\lim_{\varepsilon \to 0} Yw_{0,\varepsilon} = \overline{w}_0 \qquad \text{uniformly on the compact sets of } B. \tag{3.12}$$

Indeed, taking into account that f is locally Hölder continuous in U, and using the estimate $|Y\Gamma| \le \mathbf{c}d^{-Q}$ (which holds since Γ is δ_λ-homogeneous of degree $2-Q$ and $Y \in \mathfrak{G}_2$), we get

$$\int_{B_0} |Y\Gamma(\xi,\xi')(f_0(\xi') - f_0(\xi))|\,\mathrm{d}\xi' \le c_0 \int_{B_0} d(\xi,\xi')^{\gamma - Q} < +\infty,$$

for a suitable positive constant $c_0 = c_0(B_0)$. Moreover,

$$\begin{aligned}
|\overline{w}_0(\xi) - Yw_{0,\varepsilon}(\xi)| &= \left| \int_{B_d(\xi,2\varepsilon)} Y(\Gamma(1-\eta_\varepsilon))(\xi,\xi')(f_0(\xi') - f_0(\xi))\,\mathrm{d}\xi' \right| \\
&\le c_0 \int_{B_d(\xi,2\varepsilon)} \left(|Y\Gamma(\xi,\xi')| + \Gamma(\xi,\xi')\|\dot\eta\|_\infty |Yd(\xi,\xi')|\varepsilon^{-1}\right) d^\gamma(\xi,\xi')\,\mathrm{d}\xi' \\
&\le c_0 \int_{B_d(\xi,2\varepsilon)} (d(\xi,\xi')^{\gamma-Q} + \varepsilon^{-1} d(\xi,\xi')^{\gamma+1-Q})\,\mathrm{d}\xi' = \widetilde{c_0}\varepsilon^\gamma,
\end{aligned}$$

for every $\xi \in B$ and $\varepsilon < r/(4\beta)$, where we have used that Yd and $Y\Gamma$ are δ_λ-homogeneous functions of degree -1 and $-Q$, respectively. From (3.11) and (3.12) we have that

$$Yw_0 = \overline{w}_0 \ \text{ in } B.$$

On the other hand, from $f \in L^p(\mathbb{G})$, differentiating under the integral sign, we get

$$Yw_1(\xi) = \int_{\mathbb{G}\setminus B_0} Y\Gamma(\xi,\xi') f_1(\xi')\,\mathrm{d}\xi', \quad \xi \in B.$$

Then, taking $\xi = \xi_0$ we get

$$\begin{aligned}
Yw(\xi_0) &= Y(w_0 + w_1)(\xi_0) = \overline{w}_0(\xi_0) + Y(w_1)(\xi_0) \\
&= \int_{B_0} Y\Gamma(\xi_0,\xi')(f(\xi') - f(\xi_0))\,\mathrm{d}\xi' + \int_{\mathbb{G}\setminus B_0} Y\Gamma(\xi_0,\xi') f(\xi')\,\mathrm{d}\xi',
\end{aligned}$$

where we have used that

$$\int_{\partial B_0} \Gamma(\xi_0, \xi') < Y(\xi'), \nu(\xi') > \, \mathrm{d}\sigma(\xi') = c r^{2-Q} \int_{\partial B_0} < Y(\xi'), \nu(\xi') > \, \mathrm{d}\sigma(\xi')$$
$$= c r^{2-Q} \int_{B_0} Y(1) \, \mathrm{d}\xi' = 0.$$

This completes the proof. □

In the following proposition, by using the representation formula provided by Lemma 3.3, we get the following asymptotic estimate at 0 for the derivatives of the function w in (3.6).

Proposition 3.4. *Let $\mathbb{G}$ be a Carnot group of step two. If w is the function defined in* (3.6) *and $Y \in \mathfrak{G}_2$ is a Lie derivative of order two, then*

$$|Y w(\xi)| = \mathcal{O}(d(\xi)^{-\alpha}) \quad as \quad d(\xi) \to 0. \tag{3.13}$$

Proof. Due to (3.7) and (3.8), Lemma 3.3 applies to w in the set $U = \Omega \setminus \{0\}$. Now, let $\xi \in \Omega \setminus \{0\}$ with sufficiently small $d(\xi)$ such that $B(\xi, d(\xi)/2) \Subset \Omega \setminus \{0\}$ and let us use the representation formula (3.10) with $r = d(\xi)/2$. Taking into account that $Y\Gamma$ is a δ_λ-homogeneous function of degree $-Q$, by (3.10) we get

$$\begin{aligned} |Y w(\xi)| &\le c \int_{B_d(\xi,r)} d(\xi,\xi')^{-Q} |f(\xi') - f(\xi)| \, \mathrm{d}\xi' + c \int_{\mathbb{G} \setminus B_d(\xi,r)} d(\xi,\xi')^{-Q} |f(\xi')| \, \mathrm{d}\xi' \\ &:= c I_1(\xi) + c I_2(\xi). \end{aligned} \tag{3.14}$$

Let us begin by estimating the term $I_1(\xi)$ in (3.14). Denoted by $g := |u|^{2^*(\alpha)-2} u$, we get, for $\xi' \in B_d(\xi, r)$

$$\begin{aligned} |f(\xi') - f(\xi)| &= \left| \psi^\alpha(\xi') \frac{g(\xi')}{d(\xi')^\alpha} - \psi^\alpha(\xi) \frac{g(\xi)}{d(\xi)^\alpha} \right| \\ &\le c \left| \frac{1}{d(\xi')^\alpha} - \frac{1}{d(\xi)^\alpha} \right| + \frac{c}{d(\xi)^\alpha} |g(\xi') - g(\xi)| + \frac{c}{d(\xi)^\alpha} |\psi^\alpha(\xi') - \psi^\alpha(\xi)| \end{aligned} \tag{3.15}$$

where we have used the boundedness of ψ and g.

Now, we shall use the following inequality, whose Euclidean proof can be found e.g. in [34, Chapter 10] (see also [38], p. 574, where it is used in the Euclidean elliptic context)

$$\left| d(\xi,\eta)^{-\delta} - d(\xi',\eta)^{-\delta} \right| \le c_\gamma d(\xi,\xi')^\gamma \left(d(\xi,\eta)^{-\delta-\gamma} + d(\xi',\eta)^{-\delta-\gamma} \right)$$

where $\delta > 0$ and $0 < \gamma < 1$.

By the above inequality, and using the fact that $d(\xi)/2 \le d(\xi') \le 3/2 d(\xi)$ for $\xi' \in B_d(\xi, r)$, we get for the first term in the r.h.s. of (3.15)

$$\left| \frac{1}{d(\xi')^\alpha} - \frac{1}{d(\xi)^\alpha} \right| \le c_\gamma d(\xi,\xi')^\gamma \left(\frac{1}{d(\xi')^{\alpha+\gamma}} + \frac{1}{d(\xi)^{\alpha+\gamma}} \right) \le \frac{c_\gamma}{d(\xi)^{\alpha+\gamma}} d(\xi,\xi')^\gamma, \tag{3.16}$$

for any fixed $\gamma \in (0,1)$. Moreover, if we let $\phi := |\nabla_{\mathbb{G}} d|^2$, we get

$$|\psi^\alpha(\xi') - \psi^\alpha(\xi)| = |\phi^{\alpha/2}(\xi') - \phi^{\alpha/2}(\xi)| \le |\phi(\xi') - \phi(\xi)|^{\alpha/2}, \tag{3.17}$$

where we have used that $0 < \alpha/2 < 1$.

Now, since ϕ is smooth out of the origin, by applying the Stratified Lagrange mean value theorem (see [2, Theor. 20.3.1]), we get

$$|\phi(\xi') - \phi(\xi)| \leq c \sup_{\{\zeta\, : d(\zeta,\xi) \leq b\, d(\xi',\xi)\}} |\nabla_{\mathbb{G}}\phi(\zeta)| d(\xi, \xi') \tag{3.18}$$

where c and b are absolute constants depending only on the Carnot group and the homogeneous norm d. Hence, by (3.17) and (3.18), and taking into account that, since ϕ is smooth out of the origin and δ_λ-homogeneous of degree 0, it holds $|\nabla_{\mathbb{G}}\phi| \leq cd^{-1}$, we have

$$|\psi^\alpha(\xi') - \psi^\alpha(\xi)| \leq \frac{c}{d(\xi)^{\alpha/2}} d(\xi, \xi')^{\alpha/2}. \tag{3.19}$$

Hence, taking into account (3.16), (3.19) and the Hölder continuity of g, by (3.15) we get

$$|f(\xi') - f(\xi)| \leq \frac{c}{d(\xi)^{\alpha+\gamma}} d(\xi, \xi')^\gamma + \frac{c}{d(\xi)^\alpha} d(\xi, \xi')^\beta, \qquad \text{for all } \xi' \in B_d(\xi, r), \tag{3.20}$$

for $\gamma = \alpha/2$ and a suitable $\beta \in (0, 1)$. Hence

$$\begin{aligned} I_1(\xi) &\leq \frac{c}{d(\xi)^{\alpha+\gamma}} \int_{B_d(\xi,r)} d(\xi, \xi')^{\gamma-Q}\, \mathrm{d}\xi' + \frac{c}{d(\xi)^\alpha} \int_{B_d(\xi,r)} d(\xi, \xi')^{\beta-Q}\, \mathrm{d}\xi' \\ &\leq \frac{c}{d(\xi)^{\alpha+\gamma}} \int_0^r \rho^{\gamma-Q} \rho^{Q-1}\, \mathrm{d}\rho + \frac{c}{d(\xi)^\alpha} \int_0^r \rho^{\beta-Q} \rho^{Q-1}\, \mathrm{d}\rho \\ &= \frac{c}{d(\xi)^{\alpha+\gamma}} r^\gamma + \frac{c}{d(\xi)^\alpha} r^\beta, \end{aligned}$$

that is, letting $r = d(\xi)/2$,

$$I_1(\xi) \leq cd(\xi)^{-\alpha} + cd(\xi)^{-\alpha+\beta}. \tag{3.21}$$

Now, let us estimate the term $I_2(\xi)$ in (3.14). We observe that, being $f \in L^{\frac{Q}{\alpha},\infty}$, by applying Hölder's inequality for Lorentz spaces (see e.g. [23]), we obtain

$$I_2(\xi) \leq c \|d^{-Q}\|_{L^{\frac{Q}{Q-\alpha},1}(\mathbb{G} \setminus B_d(0,r))} \|f\|_{L^{\frac{Q}{\alpha},\infty}(\mathbb{G})} \tag{3.22}$$

Moreover, by direct calculation, one can see that $\|d^{-Q}\|_{L^{\frac{Q}{Q-\alpha},1}(\mathbb{G} \setminus B_d(0,r))} \leq cr^{-\alpha}$. Hence, by (3.22) and remembering that $r = d(\xi)/2$, we get

$$I_2(\xi) \leq cd(\xi)^{-\alpha}. \tag{3.23}$$

Finally, by using (3.21) and (3.23) in (3.14), the thesis follows. $\square$

Now, let us estimate the behavior at 0 of the second layer derivatives of the function v in (3.9). We shall apply the following estimate for the Lie derivatives of a general $\Delta_{\mathbb{G}}$-harmonic function, proved in [3]. Hereafter, we call $\Delta_{\mathbb{G}}$-*harmonic function* on an open set $U \subset \mathbb{G}$ any smooth function v such that $\Delta_{\mathbb{G}} v = 0$ in U.

Lemma 3.5. ([3, Prop. 3.7]) *Let $\mathbb{G}$ be a Carnot group and let $Y \in \mathfrak{G}_k$ be a Lie derivative of order k. If U is an arbitrary open set of $\mathbb{G}$ and v is a $\Delta_{\mathbb{G}}$-harmonic function on U, then there exists a positive constant c such that*

$$|Yv(\xi)| \le cr^{-k} \sup_{B_d(\xi,r)} |v|, \tag{3.24}$$

for every $B_d(\xi, r) \Subset U$. The constant c only depends on Y (and the structure of $\mathbb{G}$) and not on v, $r > 0$ or $\xi \in U$.

Proposition 3.6. *Let $\mathbb{G}$ be a Carnot group and let $Y \in \mathfrak{G}_2$ be a Lie derivative of order two. Let v be the function defined in* (3.9)*. Then,*

$$|Yv(\xi)| = \mathcal{O}(d(\xi)^{-2}), \quad \text{as} \;\; d(\xi) \to 0. \tag{3.25}$$

Proof. We apply the result of Lemma 3.5 to the function v in the open set $U = \Omega \setminus \{0\}$. Hence, for any $\xi \in \Omega \setminus \{0\}$ with sufficiently small $d(\xi)$, by choosing $r = d(\xi)/2$ in (3.24) and taking into account that $v \in L^\infty$, we get $|Yv(\xi)| \le cd(\xi)^{-2}$, with c not depending on ξ. □

Proof of Theorem 3.2. By (3.13) and (3.25), and recalling that $v = w - u$, the desired estimate follows. □

Remark 3.7. We finally remark that the regularity and asymptotic estimates on second layer derivatives proved in this section can be extended to weak solutions $u \in S_0^1(\Omega)$ of general singular equations of the type

$$-\Delta_{\mathbb{G}} u = \psi^\beta \frac{g}{d^\alpha}, \quad \text{in } \Omega \subset \mathbb{G},\; 0 \in \Omega, \tag{3.26}$$

where $0 < \alpha < 2$, $\beta \ge 0$ and $g \in \Gamma^\gamma_{loc}(\Omega) \cap L^s(\Omega)$, for $s = \frac{2Q}{Q+2-2\alpha}$. Only observe that here the summability exponent s is chosen to ensure that the r.h.s. of equation (3.26) belong to the space $S^{-1}(\Omega)$, i.e. the dual space of $S_0^1(\Omega)$, allowing the weak formulation of the problem. Indeed, from $d^{-\alpha} \in L^{\frac{Q}{\alpha},\infty}$ and $g \in L^{\frac{2Q}{Q+2-2\alpha}}$, by Hölder's inequality for weak L^p-spaces, it follows that $\psi^\beta \dfrac{g}{d^\alpha} \in L^{\frac{2Q}{Q+2},\frac{2Q}{Q+2-2\alpha}} = L^{(2^*)',q'}$ for a suitable $q \ge 1$. Henceforth, for any test function $\varphi \in S_0^1(\Omega)$, $\psi^\beta \dfrac{g}{d^\alpha}\varphi \in L^{1,1}(\Omega) = L^1(\Omega)$, where we have used that, if $\varphi \in S_0^1(\Omega)$, then $\varphi \in L^{2^*,q}(\Omega)$ for any $q \ge 1$ (see [37, Prop. 3.3]).

4 Pohozaev-identity and non-existence results

In this section, we provide some non existence results for solutions to pb. (1.4), sufficiently regular up to the boundary, in domains that are starshaped about the origin with respect to the dilations (2.1). To this aim, we shall use Pohozaev-type identities, modelled on the geometry of Carnot groups.

We quote that the original Pohozaev identity for the semilinear Poisson equation [39] was extended to the stratified context by Garofalo-Lanconelli [20] and Garofalo-Vassilev [21], respectively for the Heisenberg and the general Carnot case. As observed in these papers, the implementation of such identities in the subelliptic context turns out to be a very delicate task, due to the possible loss of regularity of solutions near the characteristic set of the boundary (see the definition 2.3 in Section 2).

Moreover, in the present case, new difficulties occur, due to the lack of regularity of the solutions at the origin.

Our treatment will be confined to Carnot groups of step two.

We begin by recalling some remarkable integral identities, which hold for regular functions up to the boundary (see e.g. [20], [21]). In what follows, $\mathrm{d}\sigma$ will denote the Hausdorff $(N-1)$-dimensional measure on $\partial\Omega$.

Proposition 4.1. *Let $\mathbb{G}$ be a Carnot group of step two and let $\Omega \subset \mathbb{G}$ be a C^1 bounded open set with outer normal ν. For $u \in \Gamma^2(\overline{\Omega})$, it holds*

$$\begin{aligned}\int_\Omega (-\Delta_{\mathbb{G}} u) Y u \,\mathrm{d}\xi = &-\frac{1}{2}\int_\Omega divY\,|\nabla_{\mathbb{G}} u|^2\,\mathrm{d}\xi + \sum_{i=1}^m \int_\Omega X_i u [X_i, Y] u\,\mathrm{d}\xi \\ &+ \frac{1}{2}\int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 <Y,\nu>\,\mathrm{d}\sigma - \int_{\partial\Omega}\sum_{i=1}^m X_i u <X_i,\nu> Yu\,\mathrm{d}\sigma\end{aligned} \tag{4.1}$$

where Y is any smooth vector field.

Let us specify the preceding identity by choosing $Y = Z$, where Z denotes the infinitesimal generator of the one-parameter group of dilations δ_λ, i.e. the vector field such that

$$\left[\frac{d}{d\lambda} u(\delta_\lambda(\xi))\right]_{\lambda=1} = Zu. \tag{4.2}$$

Note that, for a generic Carnot group of step r, Z has the following expression

$$Z = \sum_{i=1}^r \sum_{j=1}^{N_i} i\,\xi_j^{(i)} \frac{\partial}{\partial \xi_j^{(i)}}.$$

We recall that Z is characterized by the property that a function $u : \mathbb{G} \to \mathbb{R}$ is homogeneous of degree k with respect to δ_λ, i.e. $u(\delta_\lambda(\xi)) = \lambda^k u(\xi)$, if and only if $Zu = ku$. Moreover, the following properties hold for Z (see e.g. [12], [21]):

$$[X_i, Z] = X_i,\ \forall i = 1,\dots,m, \qquad divZ = Q. \tag{4.3}$$

By choosing $Y = Z$ in (4.1) and exploiting the above properties (4.3), we get the following identity.

Corollary 4.2. *Let $\mathbb{G}$ be a Carnot group of step two and let $\Omega \subset \mathbb{G}$ be a C^1 bounded open set. For $u \in \Gamma^2(\overline{\Omega})$, it holds*

$$\begin{aligned}\int_\Omega (-\Delta_{\mathbb{G}} u) Z u \,\mathrm{d}\xi = &-\frac{Q-2}{2}\int_\Omega |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi \\ &+ \frac{1}{2}\int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 <Z,\nu>\mathrm{d}\sigma - \int_{\partial\Omega}\sum_{i=1}^m X_i u <X_i,\nu> Zu\,\mathrm{d}\sigma.\end{aligned} \tag{4.4}$$

Moreover, if in addition $u = 0$ on $\partial\Omega$, identity (4.4) can be rewritten as follows.

Corollary 4.3. *Let $u \in \Gamma^2(\overline{\Omega})$ and assume that $u = 0$ on $\partial\Omega$. There holds*

$$\int_\Omega (-\Delta_{\mathbb{G}} u) Z u \,\mathrm{d}\xi = -\frac{Q-2}{2}\int_\Omega |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi - \frac{1}{2}\int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 <Z,\nu>\mathrm{d}\,\sigma. \tag{4.5}$$

Proof. Since $u = 0$ on $\partial\Omega$, then $\nabla u = \dfrac{\partial u}{\partial \nu}\nu$ on $\partial\Omega$. This gives

$$\sum_{i=1}^{m} X_i u <X_i, \nu> Zu = \frac{\partial u}{\partial \nu}\sum_{i=1}^{m} X_i u <X_i, \nu><Z, \nu>= |\nabla_{\mathbb{G}} u|^2 <Z, \nu> \qquad \text{on } \partial\Omega.$$

Substituting the above relation in (4.4), the thesis follows. $\square$

Let us now recall the definition of δ_λ-starshaped domain.

Definition 4.4. *Let $\Omega \subset \mathbb{G}$ be a C^1 connected open set, containing 0 at its interior. We say that Ω is δ_λ-starshaped with respect to the origin if*

$$<Z, \nu> (\xi) \geq 0 \quad \forall \xi \in \partial\Omega.$$

Now we prove a non-existence result for nonnegative weak solutions to pb. (1.4) assuming *a priori* Γ^2-regularity (out of the origin) up to the boundary of Ω.

Theorem 4.5. *Let $\mathbb{G}$ be a Carnot group of step two. Let $\Omega \subset \mathbb{G}$ be a smooth connected bounded domain, δ_λ-starshaped about the origin. Then, the problem*

$$-\Delta_{\mathbb{G}} u = \psi^\alpha \frac{u^{2^*(\alpha)-1}}{d(\xi)^\alpha} \ \text{in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega \tag{4.6}$$

has no nontrivial nonnegative weak solutions $u \in S_0^1(\Omega) \cap \Gamma^2(\overline{\Omega} \setminus \{0\})$.

Proof. Let u be a weak solution of (4.6) satisfying the regularity assumptions of the Theorem. We shall begin by considering approximating domains $\Omega \setminus B_{r_n}(0)$, for an appropriate sequence of radii $r_n \to 0$. To this aim, observe that from Federer's coarea formula (see [15]), if $B_R(0)$ is a d-ball centered at 0 contained in Ω, then

$$\int_0^R \mathrm{d}s \int_{\partial B_s(0)} \left(\psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha} + |\nabla_{\mathbb{G}} u|^2\right) \frac{1}{|\nabla d|} \mathrm{d}\sigma = \int_{B_R(0)} \left(\psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha} + |\nabla_{\mathbb{G}} u|^2\right) \mathrm{d}\xi. \tag{4.7}$$

Moreover, from $u \in S_0^1(\Omega)$ and by the Hardy-Sobolev inequality (1.2), the integral in the r.h.s. of (4.7) is finite. This implies that there exists a sequence $r_n \to 0$ such that

$$r_n \int_{\partial B_{r_n}(0)} \left(\psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha} + |\nabla_{\mathbb{G}} u|^2\right) \frac{1}{|\nabla d|} \mathrm{d}\sigma \longrightarrow 0, \quad \text{as } n \to \infty. \tag{4.8}$$

Let $\Omega_{r_n} := \Omega \setminus B_{r_n}(0)$. By assumption, $u \in \Gamma^2(\overline{\Omega}_{r_n})$. Hence, identity (4.4) holds for u in Ω_{r_n}.

On the other hand, denoted by

$$f(\xi, u) := \psi^\alpha(\xi) \frac{u^{2^*(\alpha)-1}}{d(\xi)^\alpha}$$

the r.h.s. of the equation in (4.6) and letting $F(\xi,u)=\int_0^u f(\xi,t)\mathrm{d}t$, multiplying the equation by Zu and integrating by parts, we get

$$
\begin{aligned}
\int_{\Omega_{r_n}} (-\Delta_{\mathbb{G}} u)\, Zu \,\mathrm{d}\xi &= \int_{\Omega_{r_n}} f(\xi,u) Zu \,\mathrm{d}\xi \\
&= \int_{\Omega_{r_n}} Z(F(\xi,u))\,\mathrm{d}\xi - \int_{\Omega_{r_n}} Z\cdot\nabla_\xi F(\xi,u)\,\mathrm{d}\xi \\
&= -\int_{\Omega_{r_n}} divZ\, F(\xi,u)\,\mathrm{d}\xi + \int_{\partial\Omega_{r_n}} F(\xi,u) < Z,\nu > \mathrm{d}\sigma \\
&\quad - \int_{\Omega_{r_n}} Z\cdot\nabla_\xi F(\xi,u)\,\mathrm{d}\xi
\end{aligned}
\tag{4.9}
$$

Hence, combining (4.4) and (4.9), and taking into account that $divZ = Q$, we obtain

$$
\begin{aligned}
&Q\int_{\Omega_{r_n}} F(\xi,u)\,\mathrm{d}\xi + \int_{\Omega_{r_n}} Z\cdot\nabla_\xi F(\xi,u)\,\mathrm{d}\xi - \frac{Q-2}{2}\int_{\Omega_{r_n}} |\nabla_{\mathbb{G}} u|^2\,\mathrm{d}\xi \\
&\quad = \int_{\partial\Omega_{r_n}} \sum_{i=1}^m X_i u < X_i,\nu > Zu\,\mathrm{d}\sigma - \frac{1}{2}\int_{\partial\Omega_{r_n}} |\nabla_{\mathbb{G}} u|^2 < Z,\nu > \mathrm{d}\sigma \\
&\quad + \int_{\partial\Omega_{r_n}} F(\xi,u) < Z,\nu > \mathrm{d}\sigma.
\end{aligned}
\tag{4.10}
$$

Now, let $r_n \to 0$ in the identity (4.10). Taking into account that

$$
F(\xi,u) = \frac{1}{2^*(\alpha)}\psi^\alpha(\xi)\frac{u^{2^*(\alpha)}}{d(\xi)^\alpha} \tag{4.11}
$$

and due to the integrability of the functions $F(\xi,u)$ and $|\nabla_{\mathbb{G}} u|^2$, we get

$$
Q\int_{\Omega_{r_n}} F(\xi,u)\,\mathrm{d}\xi - \frac{Q-2}{2}\int_{\Omega_{r_n}} |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi \longrightarrow Q\int_{\Omega} F(\xi,u)\,\mathrm{d}\xi - \frac{Q-2}{2}\int_{\Omega} |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi \tag{4.12}
$$

as $r_n \to 0$. Moreover, computing the second integral in the l.h.s of (4.10), we get

$$
\begin{aligned}
\int_{\Omega_{r_n}} Z\cdot\nabla_\xi F(\xi,u)\,\mathrm{d}\xi &= \frac{1}{2^*(\alpha)}\int_{\Omega_{r_n}} Z\psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha}\,\mathrm{d}\xi - \frac{\alpha}{2^*(\alpha)}\int_{\Omega_{r_n}} \psi^\alpha d^{-\alpha-1} Zd\, u^{2^*(\alpha)}\,\mathrm{d}\xi \\
&= -\frac{\alpha}{2^*(\alpha)}\int_{\Omega_{r_n}} \psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha}\,\mathrm{d}\xi,
\end{aligned}
$$

where we have used that $Z\psi = 0$ and $Zd = d$, since they are δ_λ-homogeneous functions, respectively of degree zero and one. Therefore, again from the integrability of the function $\psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha}$, we have

$$
\int_{\Omega_{r_n}} Z\cdot\nabla_\xi F(\xi,u)\,\mathrm{d}\xi \longrightarrow \int_{\Omega} Z\cdot\nabla_\xi F(\xi,u)\,\mathrm{d}\xi, \qquad \text{as } r_n \to 0. \tag{4.13}
$$

Let us now prove that the boundary integrals in (4.10) vanish on ∂B_{r_n}, as $r_n \to 0$. Observe that, since $\nu = -\dfrac{\nabla d}{|\nabla d|}$ on $\partial B_{r_n}(0)$, then $< Z, \nu >= -\dfrac{Zd}{|\nabla d|} = -\dfrac{d}{|\nabla d|}$ on $\partial B_{r_n}(0)$. From this, and using (4.8), we have

$$\begin{aligned}
&\int_{\partial B_{r_n}} \left(F(\xi, u) - \frac{1}{2}|\nabla_{\mathbb{G}} u|^2 \right) | < Z, \nu > | \, \mathrm{d}\sigma \\
&\qquad = r_n \int_{\partial B_{r_n}} \left(\frac{1}{2^*(\alpha)} \psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha} - \frac{1}{2}|\nabla_{\mathbb{G}} u|^2 \right) \frac{1}{|\nabla d|} \, \mathrm{d}\sigma \longrightarrow 0, \qquad \text{as } r_n \to 0.
\end{aligned} \tag{4.14}$$

Concerning the remaining boundary integral $\int_{\partial B_{r_n}} \sum_{i=1}^m X_i u < X_i, \nu > Zu \, \mathrm{d}\sigma$, let us begin by observing that, since $\nu = -\frac{\nabla d}{|\nabla d|}$ on ∂B_{r_n} and $|\nabla_{\mathbb{G}} d|$ is bounded, then

$$|X_i u < X_i, \nu > | \le \frac{|X_i u \cdot X_i d|}{|\nabla d|} \le c \frac{|\nabla_{\mathbb{G}} u|}{|\nabla d|} \quad \text{on } \partial B_{r_n}. \tag{4.15}$$

Moreover, since $\mathbb{G}$ is a step two Carnot group, denoted by n the dimension of the second layer of the group and by $\xi = (x, t) \in \mathbb{R}^m \times \mathbb{R}^n$ any point of $\mathbb{G}$, we have

$$|Zu| = \left| \sum_{j=1}^m x_j \frac{\partial u}{\partial x_j} + \sum_{j=1}^n 2t_j \frac{\partial u}{\partial t_j} \right| \le c \left(d|\nabla_{\mathbb{G}} u| + d^2 |\nabla_t u| \right) \quad \text{in } \Omega \setminus \{0\}. \tag{4.16}$$

Here we have used that the vector fields X_j's have the form $X_j = \partial_{x_j} + \frac{1}{2} \sum_{k=1}^n \sum_{i=1}^m B_{j,i}^{(k)} x_i \partial_{t_k}$, where the $B^{(k)}$'s are linearly independent skew-symmetric $m \times m$ matrices (see e.g. [2, Chapter 3]). Hence, taking into account that the functions x_j and t_j are smooth and δ_λ-homogeneous, respectively of degree one and two, the estimate (4.16) follows.

Now, by Theorem 3.2 we know that

$$|\nabla_t u(\xi)| = \mathcal{O}(d(\xi)^{-2}) \quad \text{as } d(\xi) \to 0. \tag{4.17}$$

Hence, by (4.15) and (4.16), and taking into account estimate (4.17), we get

$$\begin{aligned}
\left| \int_{\partial B_{r_n}} \sum_{i=1}^m X_i u < X_i, \nu > Zu \, \mathrm{d}\sigma \right| &\le c \left(\int_{\partial B_{r_n}} \frac{d|\nabla_{\mathbb{G}} u|^2}{|\nabla d|} \, \mathrm{d}\sigma + \int_{\partial B_{r_n}} \frac{|\nabla_{\mathbb{G}} u|}{|\nabla d|} \, \mathrm{d}\sigma \right) \\
&\le c r_n \int_{\partial B_{r_n}} \frac{|\nabla_{\mathbb{G}} u|^2}{|\nabla d|} \, \mathrm{d}\sigma + c \left(\int_{\partial B_{r_n}} \frac{|\nabla_{\mathbb{G}} u|}{|\nabla d|} \, \mathrm{d}\sigma \right)^{1/2} \left(\int_{\partial B_{r_n}} \frac{\mathrm{d}\sigma}{|\nabla d|} \right)^{1/2}.
\end{aligned} \tag{4.18}$$

By Federer's co-area formula [15], we know that, for $r > 0$,

$$\int_{\partial B_r} \frac{\mathrm{d}\sigma}{|\nabla d|} = c_Q r^{Q-1}. \tag{4.19}$$

Hence, using (4.8) and (4.19) in the r.h.s. of (4.18), we have

$$\begin{aligned}
\left| \int_{\partial B_{r_n}} \sum_{i=1}^m X_i u < X_i, \nu > Zu \, \mathrm{d}\sigma \right| &\le o(1) + o(r_n^{-\frac{1}{2}}) r_n^{\frac{Q-1}{2}} \\
&= o(1), \quad \text{as } r_n \to 0.
\end{aligned} \tag{4.20}$$

So, taking into account (4.12), (4.13), (4.14), (4.20), and remembering that, since $u = 0$ on $\partial\Omega$, then $F(\xi, u) = 0$ on $\partial\Omega$ and $\int_{\partial\Omega} \sum_{i=1}^{m} X_i u < X_i, \nu > Zu \, \mathrm{d}\sigma = \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 < Z, \nu > \mathrm{d}\sigma$, from (4.10) we get the following identity on the whole Ω

$$Q \int_{\Omega} F(\xi, u) \, \mathrm{d}\xi + \int_{\Omega} Z \cdot \nabla_{\xi} F(\xi, u) \, \mathrm{d}\xi - \frac{Q-2}{2} \int_{\Omega} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi = \frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 < Z, \nu > \mathrm{d}\sigma \tag{4.21}$$

that is, substituting the explicit expressions of each term (see (4.11) and (4))

$$\frac{Q-\alpha}{2^*(\alpha)} \int_{\Omega} \psi^{\alpha} \frac{u^{2^*(\alpha)}}{d^{\alpha}} \, \mathrm{d}\xi - \frac{Q-2}{2} \int_{\Omega} |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi = \frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 < Z, \nu > \mathrm{d}\sigma. \tag{4.22}$$

On the other hand, by using u as a test function in (4.6), we have

$$\int_{\Omega} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi = \int_{\Omega} \psi^{\alpha} \frac{u^{2^*(\alpha)}}{d^{\alpha}} \, \mathrm{d}\xi. \tag{4.23}$$

Hence, substituting (4.23) in (4.22), and taking into account that $\frac{Q-\alpha}{2^*(\alpha)} - \frac{Q-2}{2} = 0$, we obtain

$$\int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 < Z, \nu > \mathrm{d}\sigma = 0. \tag{4.24}$$

Then, by means of a subelliptic unique continuation result (see Corollary A.1 in [20] and Corollary 10.7 in [21]), whose application requires u to be nonnegative, we can conclude that $u \equiv 0$ in Ω. We refer for the details to the analogous proofs in [20], [21]. □

Now, let us observe that, as stated in Proposition 3.1, weak solutions to pb. (1.4) fulfill the interior regularity required in Theorem 4.5. The issue of boundary regularity is, instead, more delicate, as already pointed out in the non-singular case (see [20], [21], [41]).

Indeed, unlike the Euclidean case, even for C^{∞} domains, solutions to subelliptic boundary value problems can present a loss of regularity near the boundary, due to the presence of characteristic points. This phenomenon was firstly observed by Jerison in [29]. Hence, the assumption of Γ^2-regularity up to the boundary in Theorem 4.5 is actually a strong requirement.

On the other hand, concerning non-characteristic points, Jerison [28] proved the following assertion, when $\mathbb{G} = \mathbb{H}^n$:

Let $\Omega \subset \mathbb{G}$ be a smooth bounded domain and let $f \in \Gamma^{j,\alpha}_{loc}(\overline{\Omega})$, $j \in \mathbb{N}$. For every $x_0 \in \partial\Omega$ not belonging to the characteristic set of Ω, there exists a neighborhood U of x_0 such that a solution u to $\Delta_{\mathbb{G}} u = f$ in Ω, $u = 0$ on $\partial\Omega$, belongs to $\Gamma^{j+2,\alpha}(\overline{\Omega} \cap U)$. (4.25)

An analogous result is not available for arbitrary Carnot groups, since a corresponding Schauder theory at non-characteristic points is presently lacking. However, it is reasonable to conjecture that it holds true, as observed in [21]. Hence, following analogous assumptions in [21] (see hypotesis 2.8) and [3], we shall assume its validity in what follows.

Under this hypothesis, in the next theorem we weaken the regularity assumptions of Theorem 4.5, by only requiring *a priori* boundedness of $\nabla_{\mathbb{G}} u$ and Zu (out of the origin) up to the boundary, as in [21, Theorem 3.7].

Theorem 4.6. *Let $\mathbb{G}$ be a Carnot group of step two and let $\Omega \subset \mathbb{G}$ be a connected smooth bounded domain, δ_λ-starshaped with respect to the origin, and satisfying* (3.2)*. Assume that* (4.25) *holds. Then, problem* (1.4) *has no nontrivial nonnegative solutions $u \in S^1_0(\Omega)$ such that $\nabla_{\mathbb{G}} u, Zu \in L^\infty_{loc}(\overline{\Omega} \setminus \{0\})$.*

Proof. Let Σ denote the characteristic set of Ω. Recall that Σ is a compact set. Arguing as in [21], we can construct a family of exhaustion domains by means of C^∞ connected, open sets $\Omega_\varepsilon \nearrow \Omega$, as $\varepsilon \to 0$, such that $\partial\Omega_\varepsilon = \gamma^1_\varepsilon \cup \gamma^2_\varepsilon$, with $\gamma^1_\varepsilon \subset \partial\Omega \setminus \Sigma$, $\gamma^1_\varepsilon \nearrow \partial\Omega \setminus \Sigma$, $H_{N-1}(\gamma^2_\varepsilon) \to 0$.

We claim that $u \in \Gamma^2(\overline{\Omega}_\varepsilon)$. Indeed, by condition (3.2), $u \in \Gamma^\beta(\overline{\Omega})$. Moreover, the nonlinearity $f(\xi, u) = \psi^\alpha(\xi)\dfrac{u^{2^*(\alpha)-1}}{d(\xi)^\alpha}$ is $\Gamma^\gamma_{loc}(\overline{\Omega} \setminus \{0\})$ in its arguments, for some $\gamma \in (0,1)$. Then, by the assumption (4.25), for any non characteristic point $\xi_0 \in \partial\Omega$, one has $u \in \Gamma^{2,\gamma}(\overline{\Omega}_\varepsilon \cap U)$ for a suitable neighborhood U of ξ_0. Then, the claim follows.

Now, reasoning as in Theorem 4.5, on each domain Ω_ε the following identity holds

$$\begin{aligned}
&\frac{Q-\alpha}{2^*(\alpha)} \int_{\Omega_\varepsilon} \psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha}\, \mathrm{d}\xi - \frac{Q-2}{2} \int_{\Omega_\varepsilon} |\nabla_{\mathbb{G}} u|^2\, \mathrm{d}\xi - \frac{1}{2} \int_{\gamma^1_\varepsilon} |\nabla_{\mathbb{G}} u|^2 < Z, \nu > \mathrm{d}\sigma \\
&\quad = \int_{\gamma^2_\varepsilon} \sum_{i=1}^m X_i u < X_i, \nu > Zu\, \mathrm{d}\sigma - \frac{1}{2} \int_{\gamma^2_\varepsilon} |\nabla_{\mathbb{G}} u|^2 < Z, \nu > \mathrm{d}\sigma \\
&\quad\quad + \int_{\gamma^2_\varepsilon} \psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha} < Z, \nu > \mathrm{d}\sigma,
\end{aligned} \tag{4.26}$$

where we have split the boundary terms on γ^1_ε and γ^2_ε and we have used that $u = 0$ on γ^1_ε.

By the boundedness of $\nabla_{\mathbb{G}} u$ and Zu near the characteristic boundary and from the fact that $H_{N-1}(\gamma^2_\varepsilon) \to 0$, we get that the boundary integrals in the r.h.s. of (4.26) tend to zero, as $\varepsilon \to 0$.

In view of the starshapedness of Ω, the monotone convergence theorem yields

$$\int_{\gamma^1_\varepsilon} |\nabla_{\mathbb{G}} u|^2 <Z, \nu> \mathrm{d}\sigma \xrightarrow[\varepsilon \to 0]{} \int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 <Z, \nu> \mathrm{d}\sigma \tag{4.27}$$

Moreover

$$\frac{Q-\alpha}{2^*(\alpha)} \int_{\Omega_\varepsilon} \psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha}\, \mathrm{d}\xi - \frac{Q-2}{2} \int_{\Omega_\varepsilon} |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi \xrightarrow[\varepsilon \to 0]{} \frac{Q-\alpha}{2^*(\alpha)} \int_{\Omega} \psi^\alpha \frac{u^{2^*(\alpha)}}{d^\alpha}\, \mathrm{d}\xi - \frac{Q-2}{2} \int_{\Omega} |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi. \tag{4.28}$$

Therefore, by letting $\varepsilon \to 0$ in (4.26), by (4.27) and (4.28) and taking into account that the r.h.s. of (4.28) is equal to 0, we obtain as before that

$$\int_{\partial\Omega} |\nabla_{\mathbb{G}} u|^2 <Z, \nu> \mathrm{d}\sigma = 0$$

At this point, we can conclude as in Theorem 4.5. $\square$

Remark 4.7. In the non-singular case $\alpha = 0$, geometric conditions on the domain Ω ensuring the boundedness of $\nabla_{\mathbb{G}} u$ and Zu up to the boundary for solutions of problem (1.4) have been provided by Garofalo and Vassilev in [21] (see Theorems 4.6, 4.7). In addition to condition

(3.2), such domains have to be *uniformly δ_λ-starshaped* (i.e. $<Z,\nu>(\xi) \geq \beta$, for all $\xi \in \partial\Omega$, for some positive constant β) and they have to satisfy a suitable convexity condition near the characteristic set Σ, which can be stated as follows. Let $\rho \in C^\infty(\mathbb{G})$ be a defining function for Ω, i.e.

$$\Omega = \{\xi \in \mathbb{G} \,|\, \rho(\xi) < R\}$$

for some $R > 0$. Denoted by $\xi = (x,t) \in \mathbb{R}^m \times \mathbb{R}^n$ any point of a step-two Carnot group $\mathbb{G}$ and by $\Psi(\xi) := |x|^2$, the defining function ρ is required to satisfy the differential inequality

$$\Delta_{\mathbb{G}}\rho \geq c\langle \nabla_{\mathbb{G}}\rho, \nabla_{\mathbb{G}}\Psi\rangle \tag{4.29}$$

in a neighborhood U of the characteristic set Σ, for some constant $c > 0$.

The above geometric conditions are satisfied, for instance, in the Heisenberg group $\mathbb{H}^n$ by the level sets of Folland's fundamental solution, i.e. the balls defined by the gauge $d(\xi) = (|x|^4+t^2)^{1/4}$, and also by the level sets of Jerison-Lee minimizers [30], which are not functions of the gauge d. More generally, in a Carnot group of step two, they are satisfied by the d-balls defined by the homogeneous norm $d(\xi) = (|x|^4 + 16|t|^2)^{1/4}$, which is the gauge realizing Kaplan's fundamental solution in the case of H-type groups.

We note that such geometric conditions ensuring the boundedness of $\nabla_{\mathbb{G}}u$ and Zu near the boundary for a solution to pb. (1.4) in the case $\alpha = 0$ also work for the singular case $0 < \alpha < 2$, thanks to the interior Γ^2-regularity of solutions away from the origin. Therefore, extending Theorem 1.2 in [21], we can state the following.

Theorem 4.8. *Let $\mathbb{G}$ be a Carnot group of step two. For any $R > 0$, the function $u \equiv 0$ is the only nonnegative weak solution of* (1.4) *in the d-ball $B_d(0,R)$ defined by the homogeneous norm $d(\xi) = (|x|^4 + 16|t|^2)^{1/4}$.*

5 Qualitative properties of Hardy-Sobolev extremals

For any open set $\Omega \subset \mathbb{G}$, let

$$S_\alpha(\Omega) := \inf_{u \in S_0^1(\Omega)} \frac{\int_\Omega |\nabla_{\mathbb{G}}u|^2 \,\mathrm{d}\xi}{\left(\int_\Omega \psi^\alpha \frac{|u|^{2^*(\alpha)}}{d(\xi)^\alpha}\mathrm{d}\xi\right)^{2/2^*(\alpha)}}. \tag{5.1}$$

Thanks to the Hardy-Sobolev inequality (1.2), the value $S_\alpha(\Omega)$ turns out to be positive. Moreover, due to the dilation invariance of the ratio in (5.1) with respect to the rescalings (1.3), if $0 \in \Omega$, then $S_\alpha(\Omega)$ is independent of Ω and $S_\alpha(\Omega) = S_\alpha(\mathbb{G})$. In the sequel the best constant $S_\alpha(\mathbb{G})$ will be simply denoted by S_α.

Let us now discuss the existence of minimizers for the best constant $S_\alpha = S_\alpha(\mathbb{G})$, namely the existence of extremal functions for the Hardy Sobolev inequality (1.2).

In the case $\alpha = 0$, the existence of Sobolev extremals in the general Carnot case has been obtained by Garofalo and Vassilev [21] by means of a suitable adaptation of Lions' concentration compactness method [35] (recall that such extremals are explicitly known only in the Heisenberg group case, where they have been computed by Jerison and Lee in [27]). We also quote that Lions' method has been used by Vassilev [41] to get existence of Sobolev extremals in the quasilinear Carnot case.

In the singular case, i.e. when $0 < \alpha < 2$, the existence of Hardy-Sobolev extremals has been proved by Han and Niu in [24], in the general quasilinear case, for the subclass of the H-type groups. We extend this result for general Carnot groups in the semiliner case under consideration.

Theorem 5.1. *Let $\mathbb{G}$ be a Carnot group, $Q > 2$. Then, the infimum in the extremal problem*

$$S_\alpha = \inf \left\{ \int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \,\Big|\, u \in S^1(\mathbb{G}),\, \int_{\mathbb{G}} \psi^\alpha \frac{|u|^{2^*(\alpha)}}{d(\xi)^\alpha} \,\mathrm{d}\xi = 1,\, 0 \le \alpha < 2 \right\} \tag{5.2}$$

is attained at some function $u \in S^1(\mathbb{G})$.

We omit the details of the proof since it is a straightforward generalization of the arguments in [24]. We only quote here the two compactness lemmas which are the main ingredients of the proof.

Lemma 5.2. *(The first Concentration-Compactness Lemma). Let $\{u_n\} \subset S^1(\mathbb{G})$ be a minimizing sequence for* (5.2)*, namely*

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u_n|^2 \,\mathrm{d}\xi \longrightarrow S_\alpha, \quad \int_{\mathbb{G}} \psi^\alpha \frac{|u_n|^{2^*(\alpha)}}{d(\xi)^\alpha} \,\mathrm{d}\xi = 1.$$

Then, there exists a subsequence (still denoted by $\{u_n\}$) such that one of the following three alternatives holds:

(i) (Compactness) For all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ and $n_\varepsilon \ge 1$ such that

$$\int_{B_R(0)} \psi^\alpha \frac{|u_n|^{2^*(\alpha)}}{d(\xi)^\alpha} \,\mathrm{d}\xi \ge 1 - \varepsilon, \quad \forall R > R_\varepsilon, \quad \forall n > n_\varepsilon\,;$$

(ii) (Vanishing) For all $R > 0$,

$$\lim_{n\to\infty} \int_{B_R(0)} \psi^\alpha \frac{|u_n|^{2^*(\alpha)}}{d(\xi)^\alpha} \,\mathrm{d}\xi = 0;$$

(iii) (Dichotomy) There exists $\gamma \in (0,1)$ such that for all $\varepsilon > 0$ there exists $R_\varepsilon > 0$, a sequence of positive numbers $R_n \to +\infty$ and $n_\varepsilon \ge 1$ such that

$$\left| \int_{B_{R_\varepsilon}(0)} \psi^\alpha \frac{|u_n|^{2^*(\alpha)}}{d(\xi)^\alpha} \,\mathrm{d}\xi - \gamma \right| < \varepsilon,$$

$$\left| \int_{\mathbb{G}\setminus B_{R_n}(0)} \psi^\alpha \frac{|u_n|^{2^*(\alpha)}}{d(\xi)^\alpha} \,\mathrm{d}\xi \, - (1-\gamma) \right| < \varepsilon,$$

$$\int_{B_{R_n}(0)\setminus B_{R_\varepsilon}(0)} \psi^\alpha \frac{|u_n|^{2^*(\alpha)}}{d(\xi)^\alpha} \,\mathrm{d}\xi < \varepsilon$$

for any $n > n_\varepsilon$.

Lemma 5.3. *(The second Concentration-Compactness Lemma). Let $\{u_n\} \subset S^1(\mathbb{G})$ be a sequence satisfying the following condition: there exist two Radon measures μ, ν and a function $u \in S^1(\mathbb{G})$ such that*

(i) $u_n \rightharpoonup u$ *weakly in* $S^1(\mathbb{G})$*;*

(ii) $\nu_n = \psi^\alpha \dfrac{|u_n|^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi \rightharpoonup \nu$ *weakly in the sense of measures;*

(iii) $\mu_n = |\nabla_{\mathbb{G}} u_n|^2\,\mathrm{d}\xi \rightharpoonup \mu$ *weakly in the sense of measures.*

Then we have

$$\nu = \psi^\alpha \frac{|u|^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi + a\delta_{(0)}, \quad \mu \geq |\nabla_{\mathbb{G}} u|^2 \mathrm{d}\xi + b\delta_{(0)}, \quad b \geq S_\alpha a^{\frac{2}{2^*(\alpha)}},$$

where a, b are non-negative numbers and $\delta_{(0)}$ is the Dirac measure with pole at 0*.*

We point out that the explicit form of the Hardy-Sobolev extremals is not known in any Carnot group, except for the trivial Euclidean case, where they have the form

$$u_{C,\varepsilon}(x) = \frac{C}{(\varepsilon + |x|^{2-\alpha})^{\frac{n-2}{2-\alpha}}}, \qquad x \in \mathbb{R}^n,$$

for any $C, \varepsilon > 0$.

However, in the general Carnot case, we can state qualitative properties of such extremals, as we know that they are positive solutions, up to multiplicative constants, of the *limit problem* on the whole $\mathbb{G}$

$$-\Delta_{\mathbb{G}} u = \psi^\alpha \frac{u^{2^*(\alpha)-1}}{d(\xi)^\alpha}, \quad u \in S^1(\mathbb{G}). \tag{5.3}$$

Since the coefficient of the nonlinearity $a(\xi) = \frac{\psi(\xi)^\alpha}{d(\xi)^\alpha}$ belongs to the weak space $L^{\frac{Q}{\alpha},\infty}$, this problem fits in the general class of singular nonlinear problems studied by the author in [37]. In the following proposition we summarize the L_p-regularity and asymptotic properties of Hardy-Sobolev extremals we can deduce from the results in [37].

Proposition 5.4. *Let $u \in S^1(\mathbb{G})$ be an extremal function for problem* (5.2)*. Then, up to a change of sign, u is positive. Moreover*

i) $u \in L^{\frac{2^*}{2},\infty}(\mathbb{G}) \cap L^\infty(\mathbb{G})$*; hence, by interpolation,* $u \in L^p(\mathbb{G}), \forall p \in (2^*/2, +\infty]$*;*

ii) $u(\xi) \simeq d(\xi)^{2-Q}$ *as* $d(\xi) \to \infty$*.*

We only remark that the positivity property of the Hardy-Sobolev extremals follows by observing that, if u is a minimizer for S_α, then also $|u|$ is a minimizer; moreover, as a minimizer for S_α, $|u|$ is a non trivial nonnegative weak solution, up to a stretching constant, of problem (5.3). Hence, by the representation formula provided by Theorem 3.2 in [11], it follows that $|u| > 0$.

We finally emphasize that the fact that the Hardy-Sobolev extremals behave at infinity like the fundamental solution Γ of the sub-Laplacian operator is the main ingredient in the following existence proof.

6 Proof of the existence result

Let $U > 0$ be a fixed minimizer for the Hardy-Sobolev inequality (1.2) and consider, for $\varepsilon > 0$, the family of rescaled functions

$$U_\varepsilon(\xi) = \varepsilon^{\frac{2-Q}{2}} U(\delta_{\frac{1}{\varepsilon}}(\xi)).$$

The functions U_ε are solutions, up to multiplicative constants, of the equation $-\Delta_{\mathbb{G}} u = \psi^\alpha \dfrac{u^{2^*(\alpha)-1}}{d(\xi)^\alpha}$ in $\mathbb{G}$. Moreover, they satisfy

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} U_\varepsilon|^2 \mathrm{d}\xi = \int_{\mathbb{G}} \psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\, \mathrm{d}\xi = S_\alpha^{\frac{Q-\alpha}{2-\alpha}}, \quad \text{for all } \varepsilon > 0. \tag{6.1}$$

Let $R > 0$ be such that $B_d(0,R) \subset \Omega$ and let $\varphi \in C_0^\infty(B_d(0,R))$, $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_d(0,R/2)$. We define

$$u_\varepsilon(\xi) = \varphi(\xi) U_\varepsilon(\xi). \tag{6.2}$$

Lemma 6.1. *The functions u_ε satisfy the following estimates, as $\varepsilon \to 0$:*

$$\int_\Omega |\nabla_{\mathbb{G}} u_\varepsilon|^2\, \mathrm{d}\xi = S_\alpha^{\frac{Q-\alpha}{2-\alpha}} + O(\varepsilon^{Q-2}) \tag{6.3}$$

$$\int_\Omega \psi^\alpha \frac{u_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\, \mathrm{d}\xi = S_\alpha^{\frac{Q-\alpha}{2-\alpha}} + O(\varepsilon^{Q-\alpha}) \tag{6.4}$$

$$\int_\Omega u_\varepsilon^2\, \mathrm{d}\xi = \begin{cases} c\,\varepsilon^2 + O(\varepsilon^{Q-2}) & \text{if } Q > 4 \\ c\,\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & \text{if } Q = 4. \end{cases} \tag{6.5}$$

Proof. Taking into account the exact asymptotic behavior of Hardy-Sobolev extremals (see Prop. 5.4), the proof reduces to compute integrals of functions which only depend on the homogeneous distance d. Let us begin to compute

$$\begin{aligned} \int_\Omega |\nabla_{\mathbb{G}} u_\varepsilon|^2 \mathrm{d}\xi &= \int_\Omega <\nabla_{\mathbb{G}} U_\varepsilon, \nabla_{\mathbb{G}}(\varphi^2 U_\varepsilon)>\, \mathrm{d}\xi + \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 U_\varepsilon^2\, \mathrm{d}\xi \\ &= \int_\Omega \varphi^2 \psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\, \mathrm{d}\xi + \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 U_\varepsilon^2\, \mathrm{d}\xi \\ &= \int_{\mathbb{G}} \psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\, \mathrm{d}\xi + \int_\Omega |\nabla_{\mathbb{G}} \varphi|^2 U_\varepsilon^2\, \mathrm{d}\xi + \sigma(\varphi,\varepsilon), \end{aligned} \tag{6.6}$$

where

$$\sigma(\varphi,\varepsilon) = -\int_{\Omega^C} \psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\, \mathrm{d}\xi + \int_\Omega (\varphi^2 - 1)\, \psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\, \mathrm{d}\xi.$$

For the second integral in the r.h.s. of (6.6), taking into account that $\varphi \equiv 1$ on $B_{R/2}(0) =$

$B_d(0,R/2)$ and $\varphi \equiv 0$ outside of $B_R(0) = B_d(0,R)$, we get

$$
\begin{aligned}
\int_\Omega |\nabla_{\mathbb{G}}\varphi|^2 U_\varepsilon^2 \,\mathrm{d}\xi &\le c \int_{B_R(0)\setminus B_{R/2}(0)} U_\varepsilon^2(\xi)\,\mathrm{d}\xi = c\int_{B_R(0)\setminus B_{R/2}(0)} \varepsilon^{2-Q}\, U^2(\delta_{\frac{1}{\varepsilon}}\xi)\,\mathrm{d}\xi \\
&= c\varepsilon^2 \int_{B_{R/\varepsilon}(0)\setminus B_{R/2\varepsilon}(0)} U^2(\eta)\mathrm{d}\eta \\
&\le c\varepsilon^2 \int_{B_{R/\varepsilon}(0)\setminus B_{R/2\varepsilon}(0)} \frac{1}{d(\eta)^{2Q-4}}\,\mathrm{d}\eta \\
&= c\varepsilon^2 \int_{R/2\varepsilon}^{R/\varepsilon} \frac{1}{\rho^{Q-3}}\,\mathrm{d}\rho \\
&= O(\varepsilon^{Q-2}).
\end{aligned}
\tag{6.7}
$$

Moreover, it is easily seen that $\sigma(\varphi,\varepsilon) = O(\varepsilon^{Q-\alpha})$. Indeed,

$$
\begin{aligned}
0 &\le \int_\Omega \left(1-\varphi^2\right)\psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi \le \int_{B_R(0)^C} \psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi = \int_{B_{R/\varepsilon}(0)^C} \psi^\alpha \frac{U^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi \\
&\le c\int_{B_{R/\varepsilon}(0)^C} \frac{1}{d(\xi)^{2Q-\alpha}}\,\mathrm{d}\xi = c\int_{R/\varepsilon}^{+\infty} \frac{1}{\rho^{Q-\alpha+1}}\,\mathrm{d}\rho \\
&= O(\varepsilon^{Q-\alpha}),
\end{aligned}
\tag{6.8}
$$

and an analogous estimate holds for the other term in $\sigma(\varphi,\varepsilon)$. So, from (6.6), by taking into account (6.1), (6.7), (6.8), the estimate (6.3) follows.
Next, we have

$$
\begin{aligned}
\int_\Omega \psi^\alpha \frac{u_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi &= \int_\Omega \psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi + \int_\Omega (\varphi^{2^*(\alpha)}-1)\,\psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi \\
&= \int_{\mathbb{G}} \psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi - \int_{\Omega^C} \psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi + \int_\Omega (\varphi^{2^*(\alpha)}-1)\,\psi^\alpha \frac{U_\varepsilon^{2^*(\alpha)}}{d(\xi)^\alpha}\,\mathrm{d}\xi \\
&= S_\alpha^{\frac{Q-\alpha}{2-\alpha}} + O(\varepsilon^{Q-\alpha}),
\end{aligned}
$$

that is, estimate (6.4). Finally, we compute

$$
\begin{aligned}
\int_\Omega u_\varepsilon^2\,\mathrm{d}\xi = \int_\Omega \varphi^2 U_\varepsilon^2\,\mathrm{d}\xi &\ge \int_{B_{R/2}(0)} U_\varepsilon^2\,\mathrm{d}\xi = \varepsilon^2 \int_{B_{R/2\varepsilon}(0)} U^2\,\mathrm{d}\xi \\
&= \varepsilon^2\left(\int_{B_1(0)} U^2\mathrm{d}\xi + \int_{B_{R/2\varepsilon}(0)\setminus B_1(0)} U^2\mathrm{d}\xi\right) \\
&\ge c\varepsilon^2\left(1 + \int_1^{R/2\varepsilon} \frac{1}{\rho^{Q-3}}\,\mathrm{d}\rho\right) \\
&= \begin{cases} c\,\varepsilon^2 + O(\varepsilon^{Q-2}) & \text{if } Q>4 \\ c\,\varepsilon^2|\ln\varepsilon| + O(\varepsilon^2) & \text{if } Q=4, \end{cases}
\end{aligned}
$$

hence (6.5) is proved. This concludes the proof of the Lemma. □

Proof of Theorem 1.1. Following the arguments in [4], we know that a sufficient condition for the existence of a positive solution to (1.1) when $0 < \lambda < \lambda_1$ is that

$$S_{\lambda,\alpha} := \inf_{u \in S_0^1(\Omega)} Q_\lambda(u) = \inf_{u \in S_0^1(\Omega)} \frac{\int_\Omega (|\nabla_{\mathbb{G}} u|^2 - \lambda u^2)\, \mathrm{d}\xi}{\left(\int_\Omega \psi^\alpha \frac{|u|^{2^*(\alpha)}}{d(\xi)^\alpha}\, \mathrm{d}\xi\right)^{2/2^*(\alpha)}} < S_\alpha, \tag{6.9}$$

since this ensures that $S_{\lambda,\alpha}$ is achieved. In order to prove (6.9), let us compute the ratio $Q_\lambda(u)$ on the family of Sobolev concentrating functions u_ε introduced in (6.2).

From the preceding lemma, if $Q > 4$ we get

$$Q_\lambda(u_\varepsilon) \leq \frac{\left(S_\alpha^{\frac{Q-\alpha}{2-\alpha}} - c\lambda\varepsilon^2 + O(\varepsilon^{Q-2})\right)}{\left(S_\alpha^{\frac{Q-\alpha}{2-\alpha}} + O(\varepsilon^{Q-\alpha})\right)^{2/2^*(\alpha)}} = S_\alpha - c\lambda\varepsilon^2 + O(\varepsilon^{Q-2}) < S_\alpha,$$

for any $\lambda > 0$ and $\varepsilon > 0$ sufficiently small. Similarly, if $Q = 4$ we have

$$Q_\lambda(u_\varepsilon) \leq \frac{\left(S_\alpha^{\frac{Q-\alpha}{2-\alpha}} - c\lambda\varepsilon^2|\log\varepsilon| + O(\varepsilon^2)\right)}{\left(S_\alpha^{\frac{Q-\alpha}{2-\alpha}} + O(\varepsilon^{Q-\alpha})\right)^{2/2^*(\alpha)}} = S_\alpha - c\lambda\varepsilon^2|\log\varepsilon| + O(\varepsilon^2) < S_\alpha.$$

This concludes the proof. □

Acknowledgements The author is gratefully indebted to Proff. E. Lanconelli and L. D'Ambrosio for many useful discussions and suggestions about the subject of this paper.

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