CLOSED FORMULAS FOR THE POLYHARMONIC OPERATOR UNDER SPHERICAL SYMMETRY

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Abstract. One of the simplest formulas of the calculus of several variables says that, when applied to spherical symmetric functions $u = u(\rho)$ in $\mathbb{R}^n$, the Laplace operator takes the form $\Delta u = u''(\rho) + (n-1)u'(\rho)/\rho$.

In this paper we derive the analogous explicit expression for the polyharmonic operator Δ^k in the case of spherical symmetry. Moreover, if B is a ball centered at the origin and $u \in H_0^k(B)$ is spherical symmetric, then we deduce the functional

$$J[u] = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega (\Delta^{k/2}u(x))^2\,dx & \text{if } k \text{ is even} \\ \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\Delta^{(k-1)/2}u(x)|^2\,dx & \text{if } k \text{ is odd} \end{cases}$$

of which $(-\Delta)^k$ is gradient as a sum of integrals in one variable.

The results are based upon nontrivial combinatorial identities, which are proved by means of Zeilberger's algorithm.

1. Introduction

In recent years the interest in nonlinear equations of the form $(-\Delta)^k u = f(u)$ in Ω open subset of $\mathbb{R}^n$ has considerably grown; see, for instance, the monograph [12] and the bibliography there. We can often obtain more accurate informations about these problems in the spherical symmetric case, i.e. when $\Omega = B_R = \{x \in \mathbb{R}^n : |x| < R\}$ and $u = u(\rho)$. For instance, spherical symmetry has proven to be a fundamental tool in the theory of critical dimensions for problems with critical nonlinearity, which originates from the researches of Brezis–Nirenberg [5] and Pucci–Serrin [19], and has been developed by several authors (see, among the others, [17], [8], [14], [10]).

Moreover there is a lot of problems in which spherical symmetry allows use of ODE techniqes. Among the others, we could mention:

– sharp existence and non–existence results of non–trivial solutions to the critical nonlinear equations $(-\Delta)u = \lambda u + u^5$ in three space dimensions (see [5]), $\Delta^2 u = \lambda u + u^{\frac{n+4}{n-4}}$ when $n = 5, 6, 7$ (see [7]) and $\Delta^2 u = \mu|x|^{-4}u + \lambda u + u^{\frac{n+4}{n-4}}$ (see [6]);

– existence of entire radial solutions to the equation $\Delta^2 u = f(u)$ for suitable supercritical functions f (see [11], [1], [3]), to the equation $\Delta^2 u = \lambda|x|^{-4}u + |x|^{-\beta}|u|^{q-2}u$ for suitable λ, β, q (see [4]) and to the equation $\Delta^2 u + u^{-q} = 0$ with $q > 1$ (see [16]);

– properties of nodal solutions to critical nonlinear equation $-\Delta u = \lambda u + |u|^{4/(n-2)}u$ (see [2], [9]).

2000 *Mathematics Subject Classification.* Primary 31B30; Secondary 33F10.

Key words and phrases. Polyharmonic operator, spherical symmetry, Zeilberger's algorithm.

In these papers the elementary formulas regarding spherical symmetric functions like

$$(1.1)\qquad \begin{aligned} &(k=1) \quad \Delta u = u''(\rho) + \frac{n-1}{\rho}u'(\rho)\,,\\ &(k=2) \quad \Delta^2 u = u^{(4)}(\rho) + \frac{2(n-1)}{\rho}u^{(3)}(\rho) + (n-1)(n-3)\Big(\frac{u''(\rho)}{\rho^2} - \frac{u'(\rho)}{\rho^3}\Big)\,, \end{aligned}$$

$$(1.2)\qquad \begin{aligned} &(k=1) \quad \frac{1}{2}\int_{B_R} |\nabla u(x)|^2\,dx = \frac{\omega_n}{2}\int_0^R u'(\rho)^2\rho^{n-1}\,d\rho\,,\\ &(k=2) \quad \frac{1}{2}\int_{B_R} |\Delta u(x)|^2\,dx = \frac{\omega_n}{2}\int_0^R u''(\rho)^2\rho^{n-1}\,d\rho + \frac{\omega_n}{2}(n-1)\int_0^R u'(\rho)^2\rho^{n-3}\,d\rho \end{aligned}$$

have been widely used. Hence it is natural to ask ourselves whether generalizations of (1.1)–(1.2) may be useful, and in what kind of problems. We give a couple of examples of nonlinear critical problems in which it could be very useful (not to say necessary) to know (1.1)–(1.2) for general k.

Example 1. In the work [6], where we deal with nonlinear critical problems for the biharmonic operator with Hardy potential, we study the minimum problem ($n \geq 5$)

$$(1.3)\qquad \inf_{u\in\mathcal{D}^{2,2}(\mathbb{R}^n),u\neq 0} \frac{\int_{\mathbb{R}^n} |\Delta u|^2 dx - \mu\int_{\mathbb{R}^n} |x|^{-4}u^2 dx}{(\int_{\mathbb{R}^n} |u|^{2n/(n-4)}dx)^{(n-4)/n}}\,.$$

We find that any minimiser u for (1.3) has spherical symmetry, and $u = u(\rho)$ is a multiple of the solution to the equation

$$(1.4)\qquad u^{(4)}(\rho) + \frac{2(n-1)}{\rho}u^{(3)}(\rho) + (n-1)(n-3)\Big(\frac{u''(\rho)}{\rho^2} - \frac{u'(\rho)}{\rho^3}\Big) - \mu\frac{u(\rho)}{\rho^4} = u^{\frac{n+4}{n-4}}(\rho)\,.$$

We need accurate estimates of u when $\rho \to 0$ and of u, u' when $\rho \to \infty$, so we perform a change of variable in (1.4), attaining another ODE in which the singularity ρ^{-4} disappears. If we want to use this strategy for the analogous problem for the polyharmonic operator, we need the explicit form of (1.1) for general k.

Example 2. In the same work [6], we perform a Pohozaev–type identity to get sharp non–existence results for radial solutions to the problem

$$(1.5)\qquad \Delta^2 u - \mu\frac{u}{|x|^4} = \lambda u + |u|^{\frac{4}{n-4}}u\,, \qquad u \in H_0^2(B_1)\,,$$

where B_1 is the unitary ball in $\mathbb{R}^n$. Our Pohozaev identity is obtained by comparing two admissible variations for the functional

$$(1.6)\qquad \begin{aligned} F(u) =& \frac{1}{2}\int_{B_1} |\Delta u|^2\,dx - \frac{\mu}{2}\int_{B_1}\frac{u^2}{|x|^4}\,dx - \frac{\lambda}{2}\int_{B_1} u^2\,dx - \frac{n-4}{2n}\int_{B_1}|u|^{\frac{2n}{n-4}}\,dx\\ =& \,\omega_n\Bigg(\frac{1}{2}\int_0^1 u''(\rho)^2\rho^{n-1}\,d\rho + \frac{n-1}{2}\int_0^1 u'(\rho)^2\rho^{n-3}\,d\rho\\ &-\frac{\mu}{2}\int_0^1 u(\rho)^2\rho^{n-5}\,d\rho - \frac{\lambda}{2}\int_0^1 u(\rho)^2\rho^{n-1}\,d\rho - \frac{n-4}{2n}\int_0^1 |u(\rho)|^{\frac{2n}{n-4}}\rho^{n-1}\,d\rho\Bigg)\;; \end{aligned}$$

then, non existence results follow by means of suitable radial Hardy inequalities. Again, if we want to generalize this to the polyharmonic case, we need the explicit form of (1.2) for general k.

In the light of what has been said so far, in our opinion it is worth to provide the generalization of (1.1)–(1.2) to any k: this is the purpose of the present work.

Throughout the paper $n \in \mathbb{N}$ denotes the space dimension and $k \in \mathbb{N}$ the order of the polyharmonic operator. Let us define L_n^k as the operator Δ^k in $\mathbb{R}^n$ in the spherical symmetric case; by induction on k we immediately get

$$L_n^k[u(\rho)] = u^{(2k)}(\rho) + \sum_{h=1}^{2k-1} b_{h,k}(n) \frac{u^{(2k-h)}(\rho)}{\rho^h} \tag{1.7}$$

where $b_{h,k}(n)$ are polynomials in the n variable that we want to determine; to this aim, the right choice of a basis of polynomials which makes as simple as possible the expression of the $b_{h,k}(n)$ is crucial.

Hence, let us consider the basis of polynomials in the n variable

$$P_h(n) = 2^h \left(\frac{n}{2} - \frac{1}{2}\right)^{\underline{h}} \tag{1.8}$$

where $(a)^{\underline{h}} = a(a-1)\dots(a-h+1)$ denotes the so–called *falling factorial power* (see Notations here below), and let us define the coefficients

$$c_{h,k}^j = \left(-\frac{1}{2}\right)^{h-j} \binom{j-1}{h-j} \binom{k}{j} \frac{(h-1)!}{(j-1)!}, \qquad 1 \le h \le 2k-1, \quad 1 \le j. \tag{1.9}$$

Remark 1. We have $c_{h,k}^j \neq 0 \iff \lfloor h/2 \rfloor + 1 \le j \le h \wedge k$, where $\lfloor x \rfloor$ stands for floor(x) (see Notations here below). Moreover $P_0(n) = 1$, $P_h(n) = (n-1)(n-3)\dots(n-2h+1)$.

Our first result is

Theorem 1. *Let P_j be the polynomials defined in* (1.8)*, and let $c_{h,k}^j$ be the coefficients defined in* (1.9)*. Then the following identity holds:*

$$L_n^k[u(\rho)] = u^{(2k)}(\rho) + \sum_{h=1}^{2k-1} \left(\sum_{j=\lfloor h/2 \rfloor+1}^{h \wedge k} c_{h,k}^j P_j(n) \right) \frac{u^{(2k-h)}(\rho)}{\rho^h}. \tag{1.10}$$

Remark 2. From (1.10), taking Remark 1 into account, it is evident that the polyharmonic operator takes a particularly simple form when n is *odd*. For instance, if $n = 3$, we have

$$L_3^k[u(\rho)] = u^{(2k)}(\rho) + 2k \frac{u^{(2k-1)}}{\rho}, \tag{1.11}$$

while, if $n = 5$,

$$L_5^k[u(\rho)] = u^{(2k)}(\rho) + 4k \frac{u^{(2k-1)}}{\rho} + 4k(k-1) \left(\frac{u^{(2k-2)}}{\rho^2} - \frac{u^{(2k-3)}}{\rho^3} \right). \tag{1.12}$$

This is due to the fact that, if n is odd, $P_j(n) = 0$ for $j \geq \frac{n+1}{2}$, and the sum in (1.10) is restricted at most to the first $n-1$ terms, whatever k may be. On the other hand, if n is even, all $b_{h,k}(n)$ in (1.7) are different from zero.

Remark 3. It may be proved that

$$\begin{aligned}\sum_{j=\lfloor \frac{h}{2}+1\rfloor}^{h\wedge k} c^j_{h,k} P_j(n) &= b_{h,k}(n)\\ &= \frac{1+3(-1)^h}{h\lfloor \frac{h}{2}+1\rfloor !} \left(-h\right)_{\lfloor \frac{h}{2}+1\rfloor} \left(-k\right)_{\lfloor \frac{h}{2}+1\rfloor} \left(\frac{1}{2}-\frac{n}{2}\right)_{\lfloor \frac{h}{2}+1\rfloor} \cdot \\ &\quad \cdot {}_3F_2\left[\begin{matrix}\lfloor \frac{h}{2}+1\rfloor - h, \lfloor \frac{h}{2}+1\rfloor - k, \lfloor \frac{h}{2}+1\rfloor + \frac{1}{2} - \frac{n}{2}\\ 2\lfloor \frac{h}{2}+1\rfloor - h - \frac{1}{2}, \lfloor \frac{h}{2}+1\rfloor + 1\end{matrix}; 1\right],\end{aligned} \tag{1.13}$$

where

$$ {}_3F_2\left[\begin{matrix}a, b, c\\ d, e\end{matrix}; z\right] = \sum_{j=0}^{\infty} \frac{(a)_j\,(b)_j\,(c)_j\,z^j}{(d)_j\,(e)_j\,j!} \tag{1.14}$$

is an hypergeometric function and $(m)_h = m(m+1)\dots(m+h-1)$ stands for the Pochhammer symbol (see Notations here below). We give a brief sketch of the proof of this identity.

To fix ideas, let $h = 2p-2$ be an even number, so that $\lfloor h/2+1\rfloor = p$. Being $P_j(n) = (-2)^j \left(\frac{1}{2}-\frac{n}{2}\right)_j$, we see that

$$\text{lhs}\,(1.13) = \sum_{r\geq 0} \gamma'_r \left(\frac{1}{2}-\frac{n}{2}\right)_{p+r}; \qquad \text{rhs}\,(1.13) = \sum_{r\geq 0} \gamma''_r \left(\frac{1}{2}-\frac{n}{2}\right)_{p+r} \tag{1.15}$$

and we want to check that $\gamma'_r = \gamma''_r$. This can be achieved by observing at first that

$$\begin{aligned}\gamma'_0 &= (-2)^p c^p_{2p-2,k} = 4\binom{k}{p}\frac{(2p-3)!}{(p-2)!},\\ \gamma''_0 &= \frac{2}{(p-1)p!}\,(2-2p)_p\,(-k)_p\,,\end{aligned} \tag{1.16}$$

and it is quite easy to see that $\gamma'_0 = \gamma''_0$; moreover

$$\frac{\gamma'_{r+1}}{\gamma'_r} = \frac{(-2)^{p+r+1} c^{p+r+1}_{2p-2,k}}{(-2)^{p+r} c^{p+r}_{2p-2,k}} = \frac{2(k-p-r)(-2+p-r)}{(r+1)(p+r+1)(3+2r)} \quad (r : \gamma'_r \neq 0)\,, \tag{1.17}$$

while, taking (1.14) into account,

$$\frac{\gamma''_{r+1}}{\gamma''_r} = \frac{\dfrac{(2-p)_{r+1}\,(p-k)_{r+1}}{(3/2)_{r+1}\,(p+1)_{r+1}\,(r+1)!}}{\dfrac{(2-p)_r\,(p-k)_r}{(3/2)_r\,(p+1)_r\,r!}} = \frac{2(k-p-r)(-2+p-r)}{(r+1)(p+r+1)(3+2r)} \quad (r : \gamma''_r \neq 0)\,. \tag{1.18}$$

Being $\gamma_0' = \gamma_0''$, by (1.17) and (1.18) we get that $\gamma_r' = \gamma_r''$, so that (1.13) holds true for h even. The reasonment is quite analogous for h odd.

However identity (1.13) has little practical benefit, so we shall go into this no more.

The second result we present here concerns, when spherical symmetry occours, a representation formula for the functional of which the polyharmonic operator is gradient as a sum of simple integrals in one variable. As usual, we shall denote by ω_n the $(n-1)$–dimensional measure of $\mathbb{S}^n = \{x : |x| = 1\}$.

Let $\Omega \subset \mathbb{R}^n$ be an open set, and let us consider the functional

$$J_n^k[u] = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega (\Delta^{k/2}u(x))^2\,dx & \text{if } k \text{ is even} \\ \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\Delta^{(k-1)/2}u(x)|^2\,dx & \text{if } k \text{ is odd} \end{cases} \tag{1.19}$$

of which $(-\Delta)^k$ is gradient; integration by parts brings at once to

$$J_n^k[u] = \frac{1}{2}\int_\Omega u(x)(-\Delta)^k u(x)\,dx \qquad \forall u \in \mathcal{D}(\Omega)\,. \tag{1.20}$$

Now let $\Omega = B_R = \{x \in \mathbb{R}^n : |x| < R\}$, $0 < R \le \infty$, and let us denote by $H^k_{0,r}(B_R)$ the space of the functions $u \in H^k_0(B_R)$ with spherical symmetry (see Notations here below). Then the following theorem holds:

Theorem 2. *Let n be an even number such that $n \ge 2k-2$, or let n be any odd number. Then, for any $u \in H^k_{0,r}(B_R)$*

$$J_n^k[u] = \frac{\omega_n}{2}\sum_{h=0}^{k-1}\frac{(k+h-1)!}{2^h h!(k-h-1)!}P_h(n)\int_0^R \big(u^{(k-h)}(\rho)\big)^2\rho^{n-1-2h}\,d\rho\,. \tag{1.21}$$

where the polynomials P_h are defined in (1.8) *and $\omega_n = \dfrac{n\pi^{n/2}}{\Gamma(1+\frac{n}{2})}$ is the $n-1$ dimensional measure of the euclidean sphere $\mathbb{S}^n$.*

Remark 4. Similarly to what happens in (1.10) (see Remark 2), the expression (1.21) takes a simpler form when n is odd, since $P_h(n) = 0$ when $h \ge \dfrac{n+1}{2}$, and hence

$$J_n^k[u] = \frac{\omega_n}{2}\sum_{h=0}^{(k-1)\wedge(n-1)/2}\frac{(k+h-1)!}{2^h h!(k-h-1)!}P_h(n)\int_0^R \big(u^{(k-h)}(\rho)\big)^2\rho^{n-1-2h}\,d\rho \qquad (n \text{ odd}). \tag{1.22}$$

Remark 5. Theorem 2 can be used, for instance, in the study of nonlinear problems for the polyharmonic operator with critical or supercritical nonlinearities, as in this kind of problems $n \ge 2k+1$.

Anyway, when n is an even number such that $2 \le n \le 2k-4$, the $\lfloor k/2 - n/4 \rfloor$ rightmost integrals in (1.21) make no more sense. In this case we can resort to an alternative representation formula for $J_n^k[u]$:

Theorem 3. *For any k, n and for any $u \in H^k_{0,r}(B_R)$ the following identity holds:*

$$J^k_n[u] = \frac{\omega_n}{2}\int_0^R \left(\left(\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{1}{\rho}\right)^{k-1} u'(\rho)\right)^2 \rho^{n+2k-3}\, d\rho\,, \tag{1.23}$$

where $\omega_n = \dfrac{n\pi^{n/2}}{\Gamma(1+\frac{n}{2})}$ *is the $n-1$ dimensional measure of the euclidean sphere $\mathbb{S}^n$.*

Remark 6. There are some differences on the use of Theorem 2 and Theorem 3. Theorem 3 holds for any dimension n, regardless to k, and the expression (1.23) looks very simple. But, indeed, it is not so manageable, because one has to compose $k-1$ times the operator $\dfrac{\mathrm{d}}{\mathrm{d}\rho}\dfrac{1}{\rho}$, which brings to more and more complicated (and not explicit) expressions when k increases, no to say that squaring the resultant expression leads to integrals involving products of different derivatives of u.

On the other hand, Theorem 2 gives a totally explicit expression for J^k_n, particularly simple for small odd space dimensions (see Remark 4). For instance, when $n = 3$, however great k may be (1.22) becomes

$$J^k_3[u] = 2\pi\left(\int_0^R \left(u^{(k)}(\rho)\right)^2 \rho^2\, d\rho + k(k-1)\int_0^R \left(u^{(k-1)}(\rho)\right)^2 d\rho\right)\,, \tag{1.24}$$

while (1.23) becomes larger and larger when k increases.

The paper is organized as follows: in Section 2, after a brief introduction about Zeilberger's algorithm, we state and prove the main combinatorial identities upon which our results are based, while in Section 3 we prove all our theorems.

Notations

$\binom{m}{h}$ *Binomial coefficient.* For integers m, h, $m \geq 0$, it is defined as

$$\binom{m}{h} = \begin{cases} \dfrac{m!}{h!(m-h)!} & 0 \leq h \leq m\,; \\ 0 & h < 0 \text{ or } h > m\,. \end{cases}$$

Moreover, when we have an expression of the form $Expr = \binom{m}{h} \times Expr1$ and $h < 0$ or $h > m$, then we mean that $Expr = 0$ whatever may be *Expr1*.

$(a)^{\underline{m}}$ *Falling factorial power* (see [13]). For $a \in \mathbb{R}$ and m non negative integer it is defined as

$$(a)^{\underline{m}} = \begin{cases} \displaystyle\prod_{j=0}^{m-1}(a-j) & m \geq 1\,; \\ 1 & m = 0\,. \end{cases}$$

$(a)_m$ *Pochhammer symbol.* For $a \in \mathbb{R}$ and m non negative integer it is defined as

$$(a)_m = \begin{cases} \prod_{j=0}^{m-1} (a+j) & m \geq 1\,; \\ 1 & m = 0\,. \end{cases}$$

Sometimes $(a)_m$ is also called *rising factorial power*, and it is denoted by $(a)^{\overline{m}}$ (see [13]). Moreover, let us remark that $(a)^{\underline{m}} = (-1)^m (-a)_m$.

$\lfloor x \rfloor$ *Floor function.* For $x \in \mathbb{R}$ it is the greatest integer less or equal than x.

B_R The open ball $\{x \in \mathbb{R}^n : |x| < R\}$; $B_\infty = \mathbb{R}^n$.

$\mathcal{D}(\Omega)$ C^∞ functions with compact support in Ω.

$\mathcal{D}_r(B_R)$ Spherically symmetric functions in $\mathcal{D}(B_R)$. Let us remark that, if $u(\rho) \in \mathcal{D}_r(B_R)$, then $u^{(h)}(0) = 0$ for any h odd.

$H^k_{0,r}(B_R)$ The completion of $\mathcal{D}_r(B_R)$ in the H^k norm.

$O(\rho^h)$ We say that $f(\rho) = O(\rho^h)$ iff $\exists\, \delta > 0 : f(\rho)\rho^{-h}$ is bounded for $\rho \in (0, \delta)$.

lhs (X.Y)
rhs (X.Y) Left hand side, right hand side of formula (X.Y).

2. A couple of identities obtained via Zeilberger's algorithm

Zeilberger's algorithm is a smart procedure which allows in a great number of situations to obtain a closed formula for a finite sum like

$$\sigma(k) = \sum_{h=0}^{p} F(k, h)\,, \tag{2.1}$$

where $F(k; h)$ is a hypergeometric term in both arguments, i.e. $F(k + 1, h)/F(k, h)$ and $F(k, h + 1)/F(k, h)$ are both rational functions of k, h.

Essentially, Zeilberger's algorithm consists in finding a function $G(k, h) = R(k, h)F(k, h)$ and $J + 1$ coefficients $a_j(k)$ such that

$$G(k, h + 1) - G(k, h) = \sum_{j=0}^{J} a_j(k)F(k + j, h)\,. \tag{2.2}$$

The lhs (2.2) is telescopic; suppose, moreover, that $F(k, h) = 0$ for $h < 0$ and $h > p$. Then by summing over all $h \in \mathbb{Z}$ in (2.2) we get

$$\sum_{j=0}^{J} a_j(k)\sigma(k + j) = 0 \tag{2.3}$$

i.e. we get a recursive formula for $\sigma(k)$. Obviously the simplest cases, which easily lead to a closed formula for $\sigma(k)$, are $J = 1$ or $a_j(k) = a_j$. For an exhaustive theory of Zeilberger's algorithm see [18].

The success of Zeilberger's algorithm is strictly related to the great developement of computer algebra. We used Maxima, which is part of Sage, as a computer algebra system. Zeilberger's algorithm has been implemented into Maxima by Fabrizio Caruso, and from version 5.9.3 on the package Zeilberger is part of the standard Maxima libraries. For the manual page of Zeilberger's algorithm in Maxima see [15].

In the present Section we shall determine two finite sums like (2.1) which will be crucial in the proof of our theorems. In our case the terms F in (2.1) depend also on other parameters besides k, h, but this is not relevant to Zeilberger's algorithm.

Proposition 2.1. *Let a, b, c be integers such that $1 \le a \le c-1$ and $b \le 2c$. Then*

$$\sum_h \left(-\frac{1}{2}\right)^h \binom{a}{h}\binom{a+h}{2c-b} = 4^c\, 2^{-a-1}\, \frac{a!}{(2c-1)!}\binom{2c-1}{b-1}\left(\frac{b}{2}-\frac{1}{2}\right)^{\underline{c}}\left(\frac{b}{2}-1\right)^{\overline{c-a-1}}. \tag{2.4}$$

Proof. Let us define $\sigma^{(a,c)}(b) = \text{lhs}\,(2.4)$, $\widetilde{\sigma}^{(a,c)}(b) = \text{rhs}\,(2.4)$. We want to prove that $\sigma^{(a,c)}(b) = \widetilde{\sigma}^{(a,c)}(b)$, and we shall do this by showing that both solve the problem

$$\begin{cases} y(b-2) = \dfrac{(2c-b-2a)}{(2c-b+2)}y(b) & \forall\, b \in \mathbb{Z},\ b \le 2c \\ y(2c) = 2^{-a}, \quad y(2c-1) = 0 \end{cases} \tag{2.5}$$

which obviously has an unique solution.

Let us use Zeilberger's algorithm: if we define

$$F^{(a,c)}(b,h) = \left(-\frac{1}{2}\right)^h \binom{a}{h}\binom{a+h}{2c-b}; \quad G^{(a,c)}(b,h) = F^{(a,c)}(b,h)\frac{2h(2c-b-a-h)}{2c-b+1} \tag{2.6}$$

then we get

$$G^{(a,c)}(b,h+1) - G^{(a,c)}(b,h) = (2c-b+2)F^{(a,c)}(b-2,h) - (2c-b-2a)F^{(a,c)}(b,h) \tag{2.7}$$

as both sides of (2.7) are equal to

$$\frac{h^2 - (4c-2b-2a+1)h + a^2 + a}{2c-b+1}\, F^{(a,c)}(b,h)\,.$$

The lhs (2.7) is telescopic, and $G^{(a,c)}(b,h) = 0$ for $h < 0$ or $h > a$; hence, summation over all integers h brings to

$$(2c-b+2)\sigma^{(a,c)}(b-2) - (2c-b-2a)\sigma^{(a,c)}(b) = 0 \tag{2.8}$$

and this, together with the elementary identities

$$\sigma^{(a,c)}(2c) = \sum_h \left(-\frac{1}{2}\right)^h \binom{a}{h} = 2^{-a},$$

$$\begin{aligned}\sigma^{(a,c)}(2c-1) &= \sum_h \left(-\frac{1}{2}\right)^h (a+h)\binom{a}{h} = a\sum_h \left(-\frac{1}{2}\right)^h \binom{a}{h} + \sum_h \left(-\frac{1}{2}\right)^h h\binom{a}{h} \\ &= a2^{-a} + a\sum_h \left(-\frac{1}{2}\right)^h \binom{a-1}{h-1} = a2^{-a} - a2^{-a} = 0\end{aligned} \tag{2.9}$$

shows that $\sigma^{(a,c)}$ solves (2.5).

On the other hand $\widetilde{\sigma}^{(a,c)}$ too solves (2.5); in fact a straightforward verification gives

$$\widetilde{\sigma}^{(a,c)}(b-2) = \frac{(2c-b-2a)}{(2c-b+2)}\widetilde{\sigma}^{(a,c)}(b). \tag{2.10}$$

Moreover it follows at once from the elementary identities

$$2^c\left(c-\frac{1}{2}\right)^{\underline{c}} = (2c-1)!!, \quad 2^c\,(c-1)^{\underline{c-a-1}} = 2\frac{(2c-2)!!}{a!} \tag{2.11}$$

that $\widetilde{\sigma}^{(a,c)}(2c) = 2^{-a}$, while $\widetilde{\sigma}^{(a,c)}(2c-1) = 0$, as $(c-1)^{\underline{c}} = 0$. Hence $\sigma^{(a,c)} = \widetilde{\sigma}^{(a,c)}$, and this concludes the proof. □

Corollary 2.2. *Let k, m, j be integers such that $k \geq 1$, $1 \leq m \leq 2k-1$, $1 \leq j \leq k$, and let $c^j_{h,k}$ be the coefficients defined in* (1.9)*. Then*

$$\sum_{h=1}^{2k-1}(2k-h)!\binom{m}{2k-h}c^j_{h,k} = 4^k 2^{-j}\left(\frac{m}{2}\right)^{\underline{k}}\left(\frac{m}{2}-\frac{1}{2}\right)^{\underline{k-j}}\binom{k}{j}. \tag{2.12}$$

Proof. From (1.9) we get

$$\begin{aligned}(2k-h)!\binom{m}{2k-h}c^j_{h,k} &= (2k-h)!\binom{m}{2k-h}\left(-\frac{1}{2}\right)^{h-j}\binom{j-1}{h-j}\binom{k}{j}\frac{(h-1)!}{(j-1)!} \\ =&\binom{k}{j}\frac{m!(2k-m-1)!}{(j-1)!}\left(-\frac{1}{2}\right)^{h-j}\binom{j-1}{h-j}\binom{h-1}{2k-m-1};\end{aligned} \tag{2.13}$$

hence

$$\begin{aligned}&\sum_{h=1}^{2k-1}(2k-h)!\binom{m}{2k-h}c^j_{h,k} \\ =&\binom{k}{j}\frac{m!(2k-m-1)!}{(j-1)!}\sum_h\left(-\frac{1}{2}\right)^{h-j}\binom{j-1}{h-j}\binom{h-1}{2k-m-1} \\ =&\binom{k}{j}\frac{m!(2k-m-1)!}{(j-1)!}\sum_h\left(-\frac{1}{2}\right)^{h}\binom{j-1}{h}\binom{h+j-1}{2k-m-1}\end{aligned} \tag{2.14}$$

(note that the sum has been extended to all integers h, as $\binom{j-1}{h-j} \neq 0$ iff $1 \leq j \leq h \leq 2j-1 \leq 2k-1$).

By Proposition 2.1 with $a = j-1$, $b = m+1$ and $c = k$ we get

$$\begin{aligned}
&\sum_{h=1}^{2k-1}(2k-h)!\binom{m}{2k-h}c^{j}_{h,k} \\
&=\binom{k}{j}\frac{m!(2k-m-1)!}{(j-1)!}4^k 2^{-j}\frac{(j-1)!}{(2k-1)!}\binom{2k-1}{m}\left(\frac{m}{2}\right)^{\underline{k}}\left(\frac{m}{2}-\frac{1}{2}\right)^{\underline{k-j}} \\
&=4^k 2^{-j}\left(\frac{m}{2}\right)^{\underline{k}}\left(\frac{m}{2}-\frac{1}{2}\right)^{\underline{k-j}}\binom{k}{j}.
\end{aligned} \tag{2.15}$$

□

Proposition 2.3. *Let k, m, n be integers, $k \geq 1$. Then*

$$\begin{aligned}
&\sum_{h=0}^{k-1}(-1)^h\frac{(k+h-1)!}{h!(k-h-1)!}\left(\frac{n}{2}-\frac{1}{2}\right)^{\underline{h}}(m)^{\underline{k-h}}(m+n-k-h-1)^{\underline{k-h}} \\
&= 4^k\left(\frac{m}{2}\right)^{\underline{k}}\left(\frac{m}{2}+\frac{n}{2}-1\right)^{\underline{k}}.
\end{aligned} \tag{2.16}$$

Proof. We may rewrite (2.16) as follows:

$$\begin{aligned}
&\sum_{h\geq 0}(-1)^h\binom{k-1}{h}\frac{(k+h-1)!}{(k-1)!}\left(\frac{n}{2}-\frac{1}{2}\right)^{\underline{h}}(m)^{\underline{k-h}}(m+n-k-h-1)^{\underline{k-h}} \\
&= 4^k\left(\frac{m}{2}\right)^{\underline{k}}\left(\frac{m}{2}+\frac{n}{2}-1\right)^{\underline{k}}
\end{aligned} \tag{2.17}$$

as summands in (2.17) are zero for $h \geq k$.

Just as before, let us define $\sigma^{(m,n)}(k) = \text{lhs}\,(2.17)$, $\widetilde{\sigma}^{(m,n)}(k) = \text{rhs}\,(2.17)$. We want to prove that $\sigma^{(m,n)}(k) = \widetilde{\sigma}^{(m,n)}(k)$, and we shall do this by showing that both solve the problem

$$\begin{cases} y(k+1) = (2k-m)(2k-m-n+2)y(k) & \forall\, k \geq 1 \\ y(1) = m(m+n-2) \end{cases} \tag{2.18}$$

which obviously has unique solution.

Let us use again Zeilberger's algorithm: if we define

$$\begin{aligned}
F^{(m,n)}(k,h) = (-1)^h\binom{k-1}{h}\frac{(k+h-1)!}{(k-1)!}\cdot \\
\cdot\left(\frac{n}{2}-\frac{1}{2}\right)^{\underline{h}}(m)^{\underline{k-h}}(m+n-k-h-1)^{\underline{k-h}}
\end{aligned} \tag{2.19}$$

and

$$
\begin{aligned}
G^{(m,n)}(k,h) &= 2(-1)^h \binom{k-1}{h-1} \frac{(k+h-1)!}{(k-1)!}(2+2k-m-n)\cdot \\
&\quad \cdot \left(\frac{n}{2}-\frac{1}{2}\right)^{\underline{h}} (m)^{\underline{k-h+1}} (m+n-k-h-1)^{\underline{k-h}}
\end{aligned} \tag{2.20}
$$

then

$$
G^{(m,n)}(k,h+1)-G^{(m,n)}(k,h) = F^{(m,n)}(k+1,h)-(2k-m)(2k-m-n+2)F^{(m,n)}(k,h)\,. \tag{2.21}
$$

Indeed

$$
\text{lhs}\,(2.21) = \text{rhs}\,(2.21) =
$$

$$
\begin{cases}
(-1)^{h+1}\left(2hm^2-2h(2k-n+1)m+(k-h)^2(n-1)\right)\cdot \\
\qquad \cdot\,(2+2k-m-n)\,\dfrac{(k+h-1)!}{h!(k-h)!}\cdot \\
\qquad \cdot \left(\dfrac{n}{2}-\dfrac{1}{2}\right)^{\underline{h}} (m)^{\underline{k-h}}\,(m+n-k-h-2)^{\underline{k-h-1}} \qquad (0\le h\le k-1)\,; \\
\\
(-1)^{k+1}\dfrac{(2k)!}{k!}m(2+2k-m-n)\left(\dfrac{n}{2}-\dfrac{1}{2}\right)^{\underline{k}} \qquad (h=k)\,; \\
\\
0 \qquad (h>k)\,.
\end{cases} \tag{2.22}
$$

The lhs (2.21) is telescopic, and $G^{(m,n)}(k,h)=0$ for $h\le 0$ or $h>k$; hence, summation over all integers $h\ge 0$ brings to

$$
\sigma^{(m,n)}(k+1)-(2k-m)(2k-m-n+2)\sigma^{(m,n)}(k)=0\,; \tag{2.23}
$$

moreover, being $\sigma^{(m,n)}(1)=m(m+n-2)$ we get that $\sigma^{(m,n)}(k)$ solves (2.18). On the other hand we easily see that $\widetilde{\sigma}^{(m,n)}(k)$ too solves (2.18); therefore $\sigma^{(m,n)}(k)=\widetilde{\sigma}^{(m,n)}(k)$.

□

3. Proof of the Theorems

All the proofs in the present Section are based upon the following (obvious) principle. Consider the two operators

$$
T' = \Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{2k} + \sum_{h=1}^{2k-1} \frac{a'_h}{\rho^h}\Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{2k-h}\;;\quad T'' = \Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{2k} + \sum_{h=1}^{2k-1} \frac{a''_h}{\rho^h}\Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{2k-h}\,, \tag{3.1}
$$

where a'_h and a''_h are real constants; then

$$
T'=T'' \iff T'[\rho^m]=T''[\rho^m] \quad \forall\, m: 1\le m\le 2k-1\,. \tag{3.2}
$$

Therefore, we preliminarily need to prove the following

Lemma 3.1. *For any integer m*

$$L_n^k[\rho^m] = 4^k \left(\frac{m}{2}\right)^{\underline{k}} \left(\frac{m}{2} + \frac{n}{2} - 1\right)^{\underline{k}} \rho^{m-2k} . \tag{3.3}$$

Proof. We argue by induction on k. For $k = 1$ we have

$$L_n^1[\rho^m] = \Big(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{n-1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)\rho^m = m(m+n-2)\rho^{m-2} = 4\left(\frac{m}{2}\right)^{\underline{1}} \left(\frac{m}{2} + \frac{n}{2} - 1\right)^{\underline{1}} \rho^{m-2} , \tag{3.4}$$

so (3.3) holds for $k = 1$. By the inductive hypothesis we get

$$\begin{aligned} L_n^{k+1}[\rho^m] &= L_n^k[L_n^1[\rho^m]] = L_n^k[4\left(\frac{m}{2}\right)\left(\frac{m}{2} + \frac{n}{2} - 1\right)\rho^{m-2}] \\ &= 4\left(\frac{m}{2}\right)\left(\frac{m}{2} + \frac{n}{2} - 1\right) 4^k \left(\frac{m-2}{2}\right)^{\underline{k}} \left(\frac{m-2}{2} + \frac{n}{2} - 1\right)^{\underline{k}} \rho^{m-2-2k} \\ &= 4^{k+1}\left(\frac{m}{2}\right)^{\underline{k+1}} \left(\frac{m}{2} + \frac{n}{2} - 1\right)^{\underline{k+1}} \rho^{m-2(k+1)} . \end{aligned} \tag{3.5}$$

□

Proof of Theorem 1. Taking into account Remark 1, let us define the operator

$$\widetilde{L}_n^k = \Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{2k} + \sum_{h=1}^{2k-1}\sum_{j\geq 1} c_{h,k}^j P_j(n)\frac{1}{\rho^h}\Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{2k-h} .$$

We want to prove that $L_n^k = \widetilde{L}_n^k$, so we shall use (3.2). By computing $\widetilde{L}_n^k[\rho^m]$ for $1 \leq m \leq 2k-1$ we have

$$\widetilde{L}_n^k[\rho^m] = \sum_{h=1}^{2k-1}\sum_{j=1}^{k}(2k-h)!\binom{m}{2k-h} c_{h,k}^j P_j(n)\rho^{m-2k} , \qquad 1 \leq m \leq 2k-1 ;$$

hence, by means of Corollary 2.2 and taking (1.8) into account we get

$$\begin{aligned} \widetilde{L}_n^k[\rho^m] &= 4^k \sum_{j=1}^{k} 2^{-j} \left(\frac{m}{2}\right)^{\underline{k}} \left(\frac{m}{2} - \frac{1}{2}\right)^{\underline{k-j}} \binom{k}{j} P_j(n)\rho^{m-2k} \\ &= 4^k \sum_{j=1}^{k} \left(\frac{m}{2}\right)^{\underline{k}} \left(\frac{m}{2} - \frac{1}{2}\right)^{\underline{k-j}} \left(\frac{n}{2} - \frac{1}{2}\right)^{\underline{j}} \binom{k}{j} \rho^{m-2k} . \end{aligned} \tag{3.6}$$

As $1 \leq m \leq 2k-1$, $\left(\frac{m}{2}\right)^{\underline{k}} \left(\frac{m}{2} - \frac{1}{2}\right)^{\underline{k}} = 2^{-2k}(m)^{\underline{2k}} = 0$; therefore we may extend summation in (3.6) to $j = 0$, achieving

$$\widetilde{L}_n^k[\rho^m] = 4^k \left(\frac{m}{2}\right)^{\underline{k}} \rho^{m-2k} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{m}{2} - \frac{1}{2}\right)^{\underline{k-j}} \left(\frac{n}{2} - \frac{1}{2}\right)^{\underline{j}} . \tag{3.7}$$

From the well–known binomial theorem for falling powers (see for instance [13])

$$\sum_{j=0}^{k} \binom{k}{j} (a)^{\underline{j}} (b)^{\underline{k-j}} = (a+b)^{\underline{k}}$$

and from Lemma 3.1 we finally deduce

$$\widetilde{L}_n^k[\rho^m] = 4^k \left(\frac{m}{2}\right)^{\underline{k}} \left(\frac{m}{2}+\frac{n}{2}-1\right)^{\underline{k}} \rho^{m-2k} = L_n^k[\rho^m]\,. \tag{3.8}$$

□

Now let us pass to the proofs of Theorems 2 and 3. We shall prove these theorems for $u \in \mathcal{D}_r(B_R)$; the extension to any $u \in H^k_{0,r}(B_R)$ follows by standard density argument.

Proof of Theorem 2. Let us define the operator

$$\Gamma_n^k = \rho^{1-n} \sum_{h=0}^{k-1} (-1)^h \frac{(k+h-1)!}{2^h h!(k-h-1)!} P_h(n) \Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{k-h} \left(\rho^{n-1-2h} \Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{k-h}\right)\,. \tag{3.9}$$

We want to prove that $L_n^k = \Gamma_n^k$; being Γ_n^k like the operators in (3.1), we shall use (3.2).

As $\Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^p \rho^m = (m)^{\underline{p}} \rho^{m-p}$, we have

$$\begin{aligned} &\rho^{1-n} \Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{k-h} \left(\rho^{n-1-2h} \Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{k-h} \rho^m\right) = \rho^{1-n} \Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\Big)^{k-h} (m)^{\underline{k-h}} \rho^{n-1+m-k-h} \\ &= (m)^{\underline{k-h}} (m+n-k-h-1)^{\underline{k-h}} \rho^{m-2k}\,. \end{aligned} \tag{3.10}$$

Therefore, taking (1.8), Proposition 2.3 and Lemma 3.1 into account, we get

$$\begin{aligned} \Gamma_n^k[\rho^m] &= \rho^{m-2k} \sum_{h=0}^{k-1} (-1)^h \frac{(k+h-1)!}{h!(k-h-1)!} \left(\frac{n}{2}-\frac{1}{2}\right)^{\underline{h}} (m)^{\underline{k-h}} (m+n-k-h-1)^{\underline{k-h}} \\ &= 4^k \left(\frac{m}{2}\right)^{\underline{k}} \left(\frac{m}{2}+\frac{n}{2}-1\right)^{\underline{k}} \rho^{m-2k} = L_n^k[\rho^m]\,. \end{aligned} \tag{3.11}$$

So $L_n^k = \Gamma_n^k$; hence, by (1.20) it follows

$$J_n^k[u] = \frac{\omega_n}{2} \sum_{h=0}^{k-1} \frac{(-1)^{k+h}(k+h-1)!}{2^h h!(k-h-1)!} P_h(n) \int_0^R u(\rho) \big(\rho^{n-1-2h} u^{(k-h)}(\rho)\big)^{(k-h)}\, d\rho\,. \tag{3.12}$$

It is immediate to check that, under the hypotheses of Theorem 2, integration by parts in (3.12) is allowed, leading to (1.21).

□

Before proving Theorem 3 we need some lemmas.

Lemma 3.2. *Let* $u, v \in C^1[0,R]$ *such that* $\operatorname{supp}(u) \subset [0,R)$ *and* $v(\rho) = O(\rho^2)$. *Then*

$$\int_0^R u(\rho)\left(\frac{v(\rho)}{\rho}\right)' d\rho = -\int_0^R \frac{u'(\rho)v(\rho)}{\rho}\, d\rho\,. \tag{3.13}$$

Proof. Trivial. □

Now, for any integer $h \geq 0$ let us define the operator

$$M_h = \Big(\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{1}{\rho}\Big)^h \frac{\mathrm{d}}{\mathrm{d}\rho}\,. \tag{3.14}$$

The following properties of the operators M_h are direct consequences of the definition:

$$M_h[\rho^m] = 2^{h+1}\left(\frac{m}{2}\right)^{\underline{h+1}} \rho^{m-2h-1}\,; \tag{3.15}$$

$$\begin{aligned} &u \in \mathcal{D}_r(B_R) \Longrightarrow M_h[u] = \rho\, u_h(\rho)\,, \quad u_h \in \mathcal{D}_r(B_R)\,;\\ &u \in \mathcal{D}_r(B_R),\, m \geq 2h \Longrightarrow M_h[\rho^{m+1}u] = O(\rho^{m-2h})\,; \end{aligned} \tag{3.16}$$

moreover, by (3.16) we get

$$u \in \mathcal{D}_r(B_R),\, m \geq 2h \Longrightarrow M_h[\rho^m M_h[u]] = O(\rho^{m-2h})\,. \tag{3.17}$$

Lemma 3.3. *Let* $u \in \mathcal{D}_r(B_R)$. *Then*

$$\int_0^R u(\rho)\, M_h[\rho^m\, M_h[u(\rho)]]\, d\rho = (-1)^{h+1}\int_0^R \Big(M_h[u(\rho)]\Big)^2 \rho^m\, d\rho \qquad \forall\, m \geq 2h\,. \tag{3.18}$$

Proof. We argue by induction on h. If $h = 0$, (3.18) becomes

$$\int_0^R u(\rho)\,(\rho^m\, u'(\rho))'\, d\rho = -\int_0^R u'(\rho)^2\, \rho^m\, d\rho \tag{3.19}$$

which obviously holds true. So let us suppose that (3.18) holds true for a certain h; we want to prove that it holds true also for $h+1$. Let us define $w(\rho) = u'(\rho)/\rho$; then $w \in \mathcal{D}_r(B_R)$ and $M_{h+1}[u] = M_h[w]$. Moreover, let $m \geq 2h+2$; then

$$\int_0^R u(\rho)\, M_{h+1}[\rho^m\, M_{h+1}[u(\rho)]]\, d\rho = \int_0^R u(\rho)\,\frac{\mathrm{d}}{\mathrm{d}\rho}\Big(\frac{1}{\rho}M_h[\rho^m\, M_h[w(\rho)]]\Big)\, d\rho\,. \tag{3.20}$$

Let us define $v(\rho) = M_h[\rho^m\, M_h[w(\rho)]]$; then by (3.17), being $m \geq 2h+2$, we get $v(\rho) = O(\rho^2)$; therefore by Lemma 3.2, (3.20) and the inductive hypothesis we obtain

$$\begin{aligned} &\int_0^R u(\rho)\, M_{h+1}[\rho^m\, M_{h+1}[u(\rho)]]\, d\rho = -\int_0^R w(\rho)\, M_h[\rho^m\, M_h[w(\rho)]]\, d\rho\\ =&(-1)^{h+2}\int_0^R \Big(M_h[w(\rho)]\Big)^2 \rho^m\, d\rho = (-1)^{h+2}\int_0^R \Big(M_{h+1}[u(\rho)]\Big)^2 \rho^m\, d\rho\,. \end{aligned} \tag{3.21}$$

□

Proof of Theorem 3. Let us define the operator

$$\Phi_n^k = \rho^{1-n} M_{k-1} \circ (\rho^{n+2k-3} M_{k-1}) \,. \tag{3.22}$$

We want to prove that $L_n^k = \Phi_n^k$; being Φ_n^k like the operators in (3.1), we shall use (3.2). From (3.15) we obtain

$$\begin{aligned} \Phi_n^k[\rho^m] &= \rho^{1-n} M_{k-1}[\rho^{n+2k-3} M_{k-1}[\rho^m]] = 2^k \left(\frac{m}{2}\right)^{\underline{k}} \rho^{1-n} M_{k-1}[\rho^{n+m-2}] \\ &= 4^k \left(\frac{m}{2}\right)^{\underline{k}} \left(\frac{m}{2} + \frac{n}{2} - 1\right)^{\underline{k}} \rho^{m-2k} = L_n^k[\rho^m] \,. \end{aligned} \tag{3.23}$$

So $L_n^k = \Phi_n^k$; therefore, by (1.20) and by Lemma 3.3 it follows

$$\begin{aligned} J_n^k[u] &= (-1)^k \frac{\omega_n}{2} \int_0^R u(\rho) M_{k-1}[\rho^{n+2k-3} M_{k-1}[u(\rho)]\, d\rho \\ &= \frac{\omega_n}{2} \int_0^R \Big(M_{k-1}[u(\rho)]\Big)^2 \rho^{n+2k-3}\, d\rho \\ &= \frac{\omega_n}{2} \int_0^R \left(\left(\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{1}{\rho}\right)^{k-1} u'(\rho)\right)^2 \rho^{n+2k-3}\, d\rho \,. \end{aligned} \tag{3.24}$$

□

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